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DIFFERENTIAL CALCULUS

(FOR B. A. STUDENTS)

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PREFACE TO THE FIRST EDITION

This volume is designed as an elementary text-book of Differential Calculus for the use of students preparing for the B.A. and B.Sc examinations of Indian Universities. The existing text-books in use in this country are good in their own way, but they are unsuitable, under the present conditions, as most of them are either too brief or they omit rigorous proofs on which greater stress is being laid in our Universities than us d to be the case in former years. This would form sufficient apology on the part of the authors for inflicting another text-book in a field where there already exists an extensive literature. The aim of the authors has been to give a thorough discussion, within the limits permitted in a text-book, of the fundamental principles so that the students may be enabled to attain a full comprehension of the methods of Differential Calculus and proper understanding of the spirit and power of this branch of Analysis.

The treatment is mainly based on the method of Limits, but the employment of the method of infinitesimals (method of Differentials) is indispensable for abbreviating solutions and investigations involving mechanical and geometrical applications. Consequently the laws of infinitesimals and the concepts of limits and continuity have been treated at greater length as a clear conception of these is essential if a correct use is to be made of the methods of Differential Calculus.

The authors acknowledge their great indebtedness to Dr. Leathem's Mathematical Theory of Limits, Dr. Fowler's Monograph on the Differential Geometry of Plane Curves and to de la Vallee Poussin's Infinitesimal Calculus which have been very frequently consulted in writing the theoretical portions of this book. Assistance has also been derived from the classical text-books on Differential Calculus by Todhunter, Williamson and Edwards in the preparation of the work. Majority of exercise in this book, culled from current text-books, may be regardes as the common property of all mathematicians. Quite a considerable number of these is original and the rest have been selected from the questions set at the various Universty examinations.

The proof of Rolle's Theorem under the condition that $\phi(x)$ is continuous in the interval (a, b) as given on page 92 has been specially constructed by Principal Hem Raj, M.A., and deduction of Taylor's Theorem from the Mean Value Theorem as given in the book (Art. 4, 141) is due to Professor Vidya Chandra, M.A.

The authors will feel grateful for suggestions for the improvement of the book and for notification of errors and misprints in the book.

It is hoped that there will be very few errors in the answers.

LAHORE : AUTHORS

Dated 1st December, 1932.

PREFACE TO THE FIFTH EDITION

The book has undergone a thorough revision in the present edition. In the earlier chapters on real numbers, limits and continuity, the subject matter has been entirely re-written and re-arranged. The rigorous forms of demonstration which have been the chief characteristics of this book have been maintained. A tentative definition of asymptotes on algebraic curves has been adopted in the beginning before the student is introduced to general treatment based on a more rigorous definition. We hope that this plan will appeal better to the understanding of the student. Numerous typical examples have been worked out in this book, especially in the chapters on partial differentiation, maxima minima, asymptotes, indeterminate forms and curvature and the different sets of exercises have been enriched by the inclusion of examples set in the various univesity examinations.

Bold type has been used to indicate topics of every kind for the sake of clearness and it is hoped that this will increase the usefulness of the book.

By the sad and untimely death of Principal Hem Raj we have been deprived of his valuable co-operation.

The authors will feel grateful for notification of errors and suggestions from those who use the book.

DELHI :
January, 1949. VIDYA CHANDRA

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DIFFERENTIAL CALCULUS

INTRODUCTION

REAL NUMBERS

1. Rational numbers. All positive and negative integers, zero and fractional numbers are embraced under the expression Rational Numbers. All such numbers can be written as $\frac{p}{q}$ where p and q are integers positive or negative and q is different from zero (division by zero being ruled out).

We know that the aggregate of rational numbers possesses the following two properties :—

1. It is **well ordered** ; that is to say, (i) if a and b are two different rational numbers then either a is greater than b or a is less than b, or symbolically either $a>b$ or $a<b$; and (ii) if $a>b$ and $b>c$, then $a>c$.

2. It is **dense** ; that is to say, between two rational and different numbers a and b $(a<b)$ we can always insert others $>a$ and $<b$ such that the difference between any two consecutive numbers so inserted is as small as we like.

For if $a<b$ all the numbers

$$a+\frac{b-a}{n},\ a+\frac{2(b-a)}{n},\ a+\frac{3(b-a)}{n},\ \ldots\ldots\ldots,\ a+\frac{(n-1)\ b-a)}{n},$$

where n is any positive integer lie between a and b. The difference between any two consecutive numbers is $\frac{b-a}{n}$, which can be made as small as we like by taking n sufficiently large.

2. Need for the extension of idea of numbers. Mathematical necessity demands the introduction of types of numbers different from rational numbers. The roots of positive rational numbers cannot be universally extracted by means of rational numbers. For instance, there is no rational number whose

square is 2. There is certainly no integral number whose square is equal to 2 as $1^2=1$ and all other integers have their squares $\geqslant 4$. Thus $\sqrt{2}$ could only be a fraction $\frac{p}{q}$ where p and q are prime to each other. But if $\frac{p}{q}$ is in its lowest terms, so is $\left(\frac{p}{q}\right)^2 = \frac{p}{q} \cdot \frac{p}{q}$ which, therefore, cannot reduce to the whole number 2. Hence there is no rational number whose square is equal to 2. Thus the solution of such an equation as $x^2=2$ is not possible in rational numbers.

It may be remarked, however, that the square root of 2 (and in fact of any positive rational number whatsoever) can always be found in rational numbers correct to any degree of approximation required (whether in defect or in excess).

***3. Irrational numbers.** The introduction of irrational numbers depends on the following considerations :—

Suppose that by any process whatsoever we divide all rational numbers into two classes : one lower class L and another upper class R such that every number of class L is less than every number of class R. Such a division is called a **cut**. It may be observed that if a number 'a' belongs to class L then every number less than a belongs to the same class L ; and similarly if a number 'b' belongs to class R then every number greater than b belongs to the same class R. There can arise only three cases.

(i) the lower class L contains a number n which is the greatest of all numbers in that class : so that every number less than n is in class L and every number greater than n is in class R. n separates the two classes L and R and is called the **frontier** of the two classes. We may say that 'n' corresponds to this cut.

(ii) The upper class R contains a number n which is the least of all in that class ; so that every number less than n belongs to the lower class L and every number greater than n belongs to the upper class R. In this case also n separates the two classes and is called the **frontier** of the two classes. We may say again that n corresponds to this cut.

It may be remarked that cases (i) and (ii) cannot happen simultaneously, that is, it is not possible that class L has a

greatest number n and at the same time class R has a least number n'. For, consider a number m which is greater than n and less than n' : as $m>n$, therefore, it belongs to class R, and as $m<n'$, therefore it belongs to class L, which is impossible because a number cannot belong to both the classes. It follows, therefore, that if there is a greatest number in the class L, then there is no least number in class R and if there is a least number in class R, then there is no greatest number in class L.

Both these cuts (*i*) and (*ii*) are called rational cuts. It is very easy to make such cuts. For example, all numbers less than n may be put in class L and all numbers greater than n in class R ; and to produce case (*i*) n may be put in class L and to get case (*ii*) n may be put in class R.

(*iii*) Neither the class L contains a greatest number nor the class R contains a least number. This can easily happen. For example, put all negative numbers and those positive numbers whose square is less than 2 in class L and all the rest in class R. Now there can be no greatest number in class L. For if '*a*' were a greatest number in class L, then another number greater than '*a*' could always be found by extracting the square root of 2 to a further decimal point. Similarly there can be no least number in class R. It may be observed that any other positive number n which is not a perfect square could replace 2 in the above example.

When such circumstances arise we say that the cut is irrational. There is no rational number which can serve as a frontier in this case; we introduce a new number $\sqrt{2}$ which is defined by the condition that it is greater than all numbers of class L and less than all numbers of class R. This new number is an **irrational number**. Thus every irrational cut defines an irrational number.

†**4. Real numbers.** The aggregate of rational and irrational numbers is called the aggregate of Real Numbers. It remains to show that this new aggregate is well ordered.

Let n be an irrational number defined by the cut (L, R). An irrational number n' will be equal to n if it is defined by the same cut ; that is, if it is greater than all the numbers of the class L and less than all the numbers of the class R.

Two irrational numbers n and n' are said to be different if they are defined by different cuts. It follows from this that between any two irrational but different numbers n and n' we can always insert at least one rational number (and consequently any number of them). For, let n' be defined by (L', R'). Since L, L' are not identical, it follows that at least one number k (and consequently any number of them) of L is found in R' or of L' is found in R. In any case, k is the required rational number lying between n and n'.

An irrational number $n' > n$ if it is greater than a rational number of class R and $n' < n$ if it is less than a rational number of class L.

Now we will show that between any two different real numbers n and n' we can always insert a rational number m and consequently any number of them.

Let n and n' be two real and different numbers. If both of them are rational, we have assumed the theorem known. If n, n' are both irrational we have proved the theorem, above. Let n be an irrational defined by the cut (L, R) and let n' be a rational number; either n' belongs to L or to R. If it belongs to L, then it is not the greatest number in that class so that there is a number m in L which is greater than n' and obviously less than n. Similarly if n' belongs to R then it is not the least number in that class so that let m be a number in R which is less than n' and obviously greater than n. So that m is the number inserted between n and n'.

*5. If we confine ourselves to the rational numbers only we have seen above in case (*iii*) that a cut effected in rational numbers does not always possess a frontier number. But it is not so with the aggregate of real numbers; this we will prove in the following theorem.

Dedekind's Theorem. *If by any process whatsoever a cut (L, R) be made in the aggregate of real numbers, that is to say, if we divide all the real numbers into two classes L and R such that every number of L is less than every number of R, there exists necessarily a frontier n (rational or irrational) such that every number less than n is found in class L and every number greater than n is found in class R.*

Consider only the rational numbers contained in the classes L and R. These possess a frontier n (rational or irrational), so that every rational number less than n is found in class L and every rational number greater than n is found in class R. We will now show that n serves the same purpose with respect to the irrational numbers also. This in fact is obvious, because every irrational number greater than n is also greater than an infinity of rational numbers greater than n and consequently is found in class R. Similarly an irrational number less than n is also less than an infinity of rational numbers less than n and is, therefore, to be found in class L.

6. Bounds. We will now consider of aggregates or sets (finite or infinite) of real numbers formed according to some definite laws ; for example, the set of all rational numbers between 1 and 2 (1 and 2 both excluded) or the set of all positive integers or the set of all irrational numbers lying between 3 and 5, and so on.

A set is said to be **bounded above** if a number H can be pointed out which is greater than every number of the set ; H itself is called a **rough upper bound**. It follows that if H_1 is greater than H, then H_1 also is a rough upper bound. Similarly if a number h can be pointed out which is smaller than every number of the set then the set is said to be **bounded below.** If a set is bounded above as well as below, it is said to be **bounded.**

Let us consider a set which is bounded above. With regard to this set all the real numbers can be divided into two classes L and R in the following way : all numbers which are less than at least one number of the set are put in class L and all the rest in class R. The frontier N of these two classes is called **the upper bound** of the set. It is clear that N is the least number of the class R, because the class L can have no greatest number in it. For suppose that N is the greatest number of class L ; then by the nature of formation of this class there is at leat one number (say k) in the set greater than N. Now a number N_1 can be found which is less than k and greater than N ; and as N_1 is less than at least one number of the set, therefore, N_1 belongs to class L which shows that N is not the greatest number of class L.

Since the frontier N of the two classes is the least number of class R it follows that there is no number in the set greater than N and there is at least one number in the set greater than $N-\epsilon$, where ϵ is positive and, however small, because $N-\epsilon$ belongs to class L.

Let us now consider another set which is bounded below. With regard to this set all real numbers can be divided into two classes L and R in the following way : all numbers which are greater than at least one number of the set are put in class R and all the rest in class L. The frontier n of these two classes is called **the lower bound** of the set. It is clear that n is the greatest number of the class L because the class R can have no least number in it. For suppose that n is the least number of class R ; then by the nature of formation of this class there is at least one number (say k) in the set less than n. Now a number n_1 can be found which is greater than k and less than n ; and as n is greater than at least one number k of the set, therefore, n_1 belongs to class R which shows that n is not the least number of class R.

Since the frontier n of the two classes is the greatest number of class L, it follows that there is no number in the set less than n and there is at least one number in the set less than $n+\epsilon$ where ϵ is positive and however small, because $n+\epsilon$ belongs to class R.

It follows that if a set is bounded above, its upper bound may be *defined* to be a number N such that there is no number in the set greater than N and there is at least one number in the set $>N-\epsilon$, however small positive ϵ may be.

Similarly if a set is bounded below, n is *defined to be its lower bound* if there is no number in the set less than n and there is at least one number in the set less than $n+\epsilon$.

Oscillation. If a set is bounded below as well as above it possesses both the upper bound as well as the lower bound, say N and n. The difference $N-n$ of the two bounds is called the **oscillation** *of the set.*

We now give a few examples :

Ex. 1.* Consider the set of all the positive integers from $\frac{1}{2}$ to 53.2 is the least number of the set and 53 is the greatest. The upper bound is 53 ; there is no number in the

set greater than 53 and there is at least one number (53 itself) in the set greater than $53-\epsilon$, however small ϵ may be. The lower bound of the set is 2; there is no number in the set less than 2 and there is at least one number (2 itself) in the set less than $2+\epsilon$. In this example the upper and the lower bounds are numbers in the set and we say that *the bounds are attained.*

Ex. 2. Consider the set of all numbers lying between 0 and 1, 0 and 1 not belonging to the set.

Here the upper bound is 1. There is no number in the set greater than 1 and there is at least one number (in fact an infinity of them) greater than $1-\epsilon$. Zero is the lower bound.

Here the upper as well as the lower bound does not belong to the set.

It may be remarked that if a set consists of a finite number of numbers then the greatest of them is the upper bound and the least of them is the lower bound. But if a set is infinite, then it is not necessary that it should have a greatest number or a least number. If it has a greatest number, then that is also the upper bound but if it does not possess a greatest number, then the upper bound, if it exists, is not attained. Similarly for the lower bound.

*7. Sequences.

A sequence is a set of numbers arranged according to a definite law. For example if we consider the group of all fractions whose numerator is 1 and whose denominators are positive integers, we get a set. We get a sequence if we arrange these according to some definite law, for example as $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \frac{1}{6}$ .. .. or as
$\frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{8}, \frac{1}{5}, \frac{1}{10}$ and so on.

It may be remarked that a sequence may be regarded as a set but a set is not a sequence.

A sequence is said to be bounded above if a number can be found which is greater than every number of the set. Similarly a sequence is said to be bounded below if a number can be pointed out which is less than every number of the sequence.

8. Constants and variables.

The real numbers may be regarded as individual numbers.

When we think of them individually, we may think either of a particular specified number such as 5, $\sqrt{3}$, π or of any number, an unspecified number, the number x. This number x is called the **real variable** and the individual numbers are called the values of the variable. Thus a quantity which passes through an infinity of real values is called a real variable, whereas a quantity which remains unchanged for any particular problem is called a **constant**.

Let x be a real variable. Consider the aggregate of all the values which x is permitted to receive. If this aggregate is bounded above we say that x is bounded above; similarly if this aggregate is bounded below we say that x is bounded below. If the aggregate is bounded above as well as below, we say that x is bounded and the bounds of the aggregate are called the bounds of x. Thus x is bounded if two numbers a and b can be pointed out such that $a<x<b$.

If x takes all the values from a to b then these values are called the **range** or the **domain of the variable**. Now this range is called a **closed range**. If the end-values $x=a$, $x=b$ are included; it is then written (a, b). But if the end-value $x=a$ is excluded, while $x=b$ is included, it is called **a range open at the left end**, and will be denoted by $[a, b)$ if $x=a$ is included, but $x=b$ is excluded, it is a **range open at the right end,** and is denoted by $(a, b]$ a range open at both end is called an **open range** and is denoted by (a, b).

With this notation, the aggregate of real numbers is denoted by $[-\infty, \infty]$, the positive part of the aggregate by $[0, \infty]$ and the negative part by $[-\infty, 0]$, zero being a number which is neither positive nor negative.

The symbol ∞ may be defined by the statement that $[0, \infty]$ includes every positive number, however large it may be.

When the value a only is excluded, or b only is excluded or both of them are excluded, then the ranges are sometimes denoted respectively by

$(a+0, b)$, $(a, b-0)$ and $(a+0, b-0)$.

EXAMPLES I*

Find the upper bound and the lower bound of the following sets and say whether they are attained or not :

1. All rational numbers greater than 3 and less than 5.

2. All rational numbers greater than 2 and less than or equal to 8.

3. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4} \ldots\ldots \frac{n-1}{n}$ ..... **4.** $\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots\ldots\ldots \frac{1}{n}$ .....

5. All Irrational numbers greater than 2 and less than 3.

9. Absolute value. The numerical difference betweeen x and a is denoted by $|x-a|$ or $|a-x|$ and is called the **absolute value** of the difference and is equal to $x-a$ or $a-x$ which ever is positive. The symbol $|x-a|$ is read "*modulus x minus a*" or "absolute value of x minus a ". Thus,

$|10-16| = 16-10 = 6$; $|16-10| = 16-10 = 6$;
$|0-5| = 5-0 = 5$; $|-5| = 5$; $|5| = 5$; $|0| = 0$.

Ex. 1. Prove

$$\begin{aligned}
&(i) && |x+y| \leqq |x| + |y| \\
&(ii) && |x-y| < |x| + |y| \\
&(iii) && |x-y| \geqq |x| - |y| \\
&(iv) && |xy| = |x| \times |y| \\
&(v) && \left|\frac{x}{y}\right| = |x| \div |y|, \; y \neq 0
\end{aligned}$$

Ex. 2. Show that (*i*) from $|a| < A (A > 0)$ follows $-A < a < A$ and conversely.

(*ii*) from $|a-b| < A$, $|b-c| < B$, follows $|a-c| < A+B$.

Hint : $-A < a-b < A$ and $-B < b-c < B$.

$\therefore -(A+B) < a-c < A+B$.

$\therefore |a-c| < A+B$.

Ex. 3. $|a+b+c+\ldots+K|$ is either

$= |a| + |b| + |c| + \ldots$

or $< |a| + |b| + |c| + \ldots$

Ex. 4. Show that if $|x-a| < \epsilon$ where ϵ is positive, then

(*i*) $x-\epsilon < a < x+\epsilon$ (*ii*) $a-\epsilon < x < a+\epsilon$.

[This simply means that if the difference between two numbers is less than a positive number ϵ, than if ϵ is added to any one of them, the sum becomes greater than the other number; and if ϵ is subtracted from any one of them, the difference becomes less than the other. This is obvious.]

CHAPTER I

LIMITS AND CONTINUITY

1. Limit of a variable.

Let x be a real variable which passes *successively* through an infinity of values following any law whatsoever, of such a sort that every value that x takes can be distinguished from the values which precede and those which follow and that no value of x shall be the last. We say that x *tends towards a determinate limit if the successive values of* x *approach a determinate number* a, *in such a way that the difference* $x-a$ *finally decreases in absolute value below every positive pre-assigned number, however, small.* We say then that x *has* a *for the limit* and write as

$$\textbf{Lt } x=a \text{ or } x\to a.$$

From this definition, it follows that *a variable cannot tend simultaneously towards two different limits* a *and* b $(b>a)$ for $x-a$ and $x-b$ cannot be simultaneously inferior in absolute value to $\frac{1}{2}(b-a)$.

It may be remarked that if we put $x=a+\epsilon$, then ϵ tends to zero.

The student should carefully note the following facts about the definition of a limit of a variable x :

1st. There must be a value x_μ such that $|a-x_\mu|$ is less than any pre-assigned positive number ϵ, however small.

2nd. For all succeeding values of x, difference $|a-x|$ must remain less than this number ϵ.

No more shoud be read in to the definition than is stated. For instance, *it is not stated that* a *is one of the values of* x. a **may** or **may not** be a value of x. Thus if x is assigned all real values in the interval $0<x<1$ and is increasing through these values, then Lt $x=1$ and the limit 1 is one of the values of x. Again, if x is assigned the values

$$2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}, \ldots\ldots \frac{n+1}{n} \ldots\ldots\ldots\ldots$$

we see that Lt $x=1$. But 1 is not one of the values of x. For

however large n is taken, we can take a still larger value and the corresponding value of $\frac{n+1}{n}$ is even closer to 1.

Neither should we worry (unless explicitly stated) as to how the variable must approach a.

It may approach continuously as in the 1st case or it may jump whole groups of intervening values as in the 2nd case or it may approach a through values less or greater than a.

If the values of x finally surpass *definitely* every number assignable, we say, by extension, that x has a limit infinite and tends towards $+\infty$.

This statement means that starting from some particular value, x always increases algebraically and in its increase ultimately takes and passes any value, greater than its initial value, that can be specified, however great algebraically such value may be.

In the same way, if x decreases *definitely* below every negative number assignable, we say that x has for limit $-\infty$.

It may be noted that infinity is not a number at all. Infinity by itself has no meaning for us, though phrases containing the word have meanings because of the definitions previously given for such phrases.

2. *A variable x is said to be* **Monotonic** *if it is constantly increasing (or stationary) or if it is constantly decreasing (or stationary)*; in the first case it is said to be **monotonically increasing** and in the second **monotonically decreasing**.

It is easy to show that if x is monotonic and bounded then it tends to a definite limit.

*Let the variable x be monotonically increasing and bounded above and let 'a' be its upper bound. It follows therefore that $x<a$ always and ultimately $x>a-\epsilon$, that is the difference between x and a finally becomes and remains less than any preassigned positive number ϵ, however small.

Similarly it can be shown that if x is monotonically decreasing and bounded below, then its lower bound is also its limit.

If the monotonic variable x is not bounded, we say by extension that its limit is infinite.

Hence we can say that the monotonic variable x has a limit, finite or infinite. We can also state: A monotonic sequence has a limit, finite or infinite.

3. Functions of a real variable. Two variables x and y are functions of each other in a general sense if there is some kind of dependence between the two variables. We consider one of these variables, say x, as *independent*; when a particular value is given to x, y is not arbitrary. We say y is a function of x and this is expressed by the relation

$$y=f(x)$$

It is not necessary that the functional relation between two variables may be capable of being represented by an analytical expression. Some functions are only partly represented by an analytical formula, while there are functions which are not at all so representable. For example, the number of road accidents in a belligerent country is a function of the war situation. But the functional relation cannot be expressed analytically.

The aggregate of values of x corresponding to which definite finite values of y exist are said to form the **range** or **domain of definition** or **existence** of y.

Relatively simple functions like x^m, a^x, $\sin x$ and the like of which simple properties are known may be called **elementary functions.**

A function $f(x)$ is said to be **single-valued** if it takes one and only one value for a given value of x, for example $\sin x$; a function is said to be **many valued** if it is capable of taking many values for a given value of x; such a function, for example, is $\sqrt{x}$ which takes two values of opposite signs for every positive value of x. The functions

$$y=\sin^{-1}x,\ y=\cos^{-1}x,\ y=\tan^{-1}x$$

are all many valued functions.

A function is said to be **explicit** or **implicit** according as the relation between y and x is given by an equation solved with respect to y or not solved with respect to that function.

A function is said to be **algebraic** if the relation between x and y can be expressed by an equation of which the two sides are polynomials in x and y; all other functions are called **transcendental** functions. An algebraic function is said to be **rational** if it can be expressed as a quotient of two polynomials in x, otherwise the function is **irrational**. The function is said to be **rational and integral** if it is a polynomial in x.

If $y=f(x)$ be solved for x in terms of y, we obtain $x=\phi(y)$. The functions $f(x)$ and $\phi(y)$ are said to be **inverse** *functions*.

Let $y=\frac{2x-1}{3x+2}$ then $x=\frac{2y+1}{2-3y}$, which is the inverse function.

Let $y=x^2$, then $x=\pm\sqrt{y}$; the inverse function does not exist in this case when y is negative.

4. Graphical Representation.

The variation of a function can be represented geometrically with the help of the principles of analytical geometry. If we plot on the graph all the points showing the values of y for given values of x, we get a set of points. The aggregate of all such points is called the graph of the function $y=f(x)$ and represents it. It may be remarked that graphical representation helps to enable the reader to picture the idea of the variation of a function; otherwise functions exist quite independently of any graphical representation.

5. Bounds of a function.

Let $f(x)$ be a function defined in the interval (a, b). The values of $f(x)$ form a set or aggregate. If this set has upper and lower bounds we call such bounds the bounds of the function $f(x)$ in the interval (a, b). *When both bounds exist we say that* $f(x)$ *is bounded in* (a, b). *If only the upper (or lower bound exists, we say that* $f(x)$ *is bounded above, (or below) in* (a, b).

It may be remarked that a function may have a finite value for each value of x and yet the function may not be

bounded. For example consider the following function :

$f(x)=\frac{1}{x}$, for all values of x except $x=0$

and $f(0)=0$.

For every value of x, $f(x)$ has a finite value but no number can be pointed out which cannot be exceeded by $f(x)$. For, if possible, let G be such a number.

Take $x=\frac{1}{G_1}$ where $G_1 > G$. Then for $x=\frac{1}{G_1}$, $f(x)=G_1$,

which is greater than G.

It may also be remarked that as in the case of aggregates considered beforehand, some functions may attain their bounds, sometimes only one of the bounds and sometimes none.

Oscillation. *The difference between the upper bound and the lower bound of a function is called the oscillation of the function in the interval* (a, b).

It can be easily seen that if M and m are the bounds of $f(x)$ in (a, b) and M_1 and m_1 are the bounds of $f(x)$ in another interval (c, d) wholly contained in (a, b), then $M_1 < M$ and $m_1 > m$.

6. Limit of a function.

Definition 1. *A function $f(x)$ is said to tend to the limit l as $x \to a$ if $| f(x)-l |$ finally becomes and remains less than any pre-assigned positive number, however small.*

In other words :

The function $f(x)$ is said to tend to the limit l as x tends to a, if, corresponding to any preassigned *positive number* δ however small a small positive number ϵ can always be chosen such that for all values of x for which $| x-a | < \delta$, $| f(x)-l | < \epsilon$.

Symbolically, $| f(x)-l | < \epsilon$ when $| x-a | < \delta$,

or $f(x) \to l$ when $x \to a$,

or $\underset{x \to a}{\text{Lt}}\, f(x)=l$.

It may be pointed out that δ depends upon ϵ.

It may also be remarked that $|x-a|<\delta$ involves two statements:

(*i*) If $x>a$, then $x-a<\delta$, *i. e.*, $x<a+\delta$;
(*ii*) if $x<a$, then $a-x<\delta$, *i. e.*; $x>a-\delta$

Hence x must lie between $a-\delta$ and $a+\delta$.

Ex. 1. Let us consider the function $f(x)=x^2-1$ for values of x near $x=3$.

First, let the variable x approach 3 through values less than 3, then through values greater than 3. The following table of values is obtained:—

x	x^2-1
0	−1
1	0
1·5	1·25
·2	·3
2·5	5·25
2·9	7·41
2·99	7·9401
...	..
...	...
3	8

x	x^2-1
6	35
5	24
4	15
3·5	11·25
3·1	8·61
3·01	8·0601
3·001	8·006001
...	...
...	...
3	8

We note that as x increases from 0, the function increases. When $x=3$, *the value of the function* is 8. And since the value of the function increases, it is clear that the nearer x is to 3, the nearer the value of x^2-1 is to 8 which is the limit of x^2-1.

* If we desire to find a value of x such that $f(3)-f(x)<\epsilon$ where ϵ is an arbitrarily small positive number say ·0001, *i. e.*, such that $(3^2-1)-(x^2-1)<\cdot0001$ or $9-x^2<\cdot0001$ we have only to choose x between 3 and $\sqrt{9-\cdot0001}$ as this will ensure that

$$f(3)>f(x)>f(3)-\cdot0001.$$

Thus we can find a stage in the progress of x beyond which $f(3)-f(x)<\cdot0001$ always, *i.e.*, we can bring x to a definite value x_1 sufficienty near to 3 where $x_1=\sqrt{9-\cdot0001}$ so that for all values of x between x_1 and 3, $|f(3)-f(x)|<\cdot0001$.

Hence $\delta=3-\sqrt{9-\cdot0001}$.

Hence $f(x)$ has the characteristic of approach to a limit, *viz., an endless progress of x, a corresponding progress of $f(x)$ and the impossibility of setting up a barrier number $8-\cdot0001$ between the progressing function and its limit 8.*

We therefore say that $\operatorname{Lt}_{x\to3}(x^2-1)=8$

Though in this case the *value* of $f(x)=x^2-1$ is 8 when $x=3$, it should be clearly understood that the *conception from the limit of $f(x)$ when $x\to3$, is a different conception from that of the value of $f(x)$ when $x=3$.* Of all values of x there is just one for which the corresponding value of x^2-1 does not concern us, and that is the value 3.

The limit 8 is also approached as x decreases to 3 through values >3.

These two facts are expressed by the notation

$(x^2-1)\to8$ when $x\to3$

or $\operatorname{Lt}_{x\to3}(x^2-1)=8.$

Ex. 2. Consider the function $y=f(x)=\dfrac{x^3-1}{x-1}$, and its behaviour when x draws ever nearer to the value 1.

The formula fails to specify a functional relation for $x=1$ for which it takes *the form* $\dfrac{0}{0}$ *which has no meaning and is called indeterminate.*

Although the function $\frac{x^3-1}{x-1}$ has *no definite value* for $x=1$, it has a *definite limiting value*, when $x\to 1$, as is shown below :

When $x=1{\cdot}001$, $y=\frac{x^3-1}{x-1}=3{\cdot}003001$.

When $x={\cdot}999$, $y=\frac{x^3-1}{x-1}=2{\cdot}997001$.

The figures 3·003001 and 2·997001 point to 3 as the limit of the function when $x\to 1$.

It can, however, be shown that *the limit is 3 when* $x\to 1$, without going into the details of numerical calculation.

Now $y=\frac{x^3-1}{x-1}=\frac{(x-1)(x^2+x+1)}{x-1}$.

Since $x\neq 1$, for it is $x=1$ with which we are not concerned in the present case, $\therefore$ $x-1\neq 0$, *i.e.*, $x-1$ is other than zero and, as such, it can be struck out from the numerator and denominator.

Hence for all values of x except $x=1$.

$$y=\frac{x^3-1}{x-1}=x^2+x+1.$$

This result is valid so long as x is different from 1, however near x may be to 1.

$\therefore$ $y-3=x^2+x-2=(x^2-1)+(x-1)=(x-1)(x+2)$ and $y-3$ becomes numerically smaller and smaller. We cannot choose any number, however small, below which $y-3$ cannot go.

Hence $y\to 3$ when $x\to 1$.

It should be noted that in this case

Lt $f(x)$ *has a definite value, but* $f(1)$ *has no meaning* and $x\to 1$

therefore, Lt $f(x)\neq f(1)$.
$x\to 1$

We may also proceed thus :—

We have to study the values of x approaching, but still always different from 1.

Suppose that x differs from 1 by h, h being positive or negative and being as small as we please. Put $x=1+h$ and let $h\to 0$.

We now find the limiting value of $f(x)$, *i.e.*, $f(1+h)$

$$\underset{x\to 1}{\text{Lt}}\ \frac{x^3-1}{x-1} = \underset{h\to 0}{\text{Lt}}\ \frac{(1+h)^3-1}{(1+h)-1} = \underset{h\to 0}{\text{Lt}}\ (3+3h+h^2)=3.$$

Ex. 3. Consider the behaviour of the function $y=\frac{1}{x}$ when $x\to 0$.

(*i*) *If x tends to 0 through positive values*, $\frac{1}{x}$ is positive and becomes ever greater and greater. There exists no number which $\frac{1}{x}$, *i.e.*, y cannot pass in its increase; for instance, take the number x_1 however large, then we can take

$$x<\frac{1}{x_1} \text{ in order to ensure that } y>x_1.$$

y does not, therefore, approach a limit and its behaviour is described as increase without limit. Though y has no tendency to a limit but its progress has a definite tendency which may be denoted by the symbolic statement $y\to+\infty$ ($y\to+\infty$ may be taken to mean '*y tends positively to infinity*').

(*ii*) Similarly *if x tends to 0 through negative values*, y is negative and becomes numerically larger and larger. In this case also y has no limit, but its progress is denoted by $y\to-\infty$ ($y\to-\infty$ may be read as '*y tends negatively to infinity*').

It should be noted that *neither the function is defined for $x=0$ nor does it tend to a limit when $x\to 0$.*

We now give definitions referring to other cases.

Definition II *A function $f(x)$ is said to tend to the limit l as $x\to+\infty$, if corresponding to any pre-assigned positive number ϵ, however small, a number x_0 can be found such that $|f(x)-l|<\epsilon$ for all values of $x\geqslant x_0$.*

In this case we write $\underset{x\to\infty}{\text{Lt}}\ f(x)=l$,

or $f(x)\to l$ when $x\to\infty$.

Definition III. *A function $f(x)$ is said to tend to $+\infty$ with x if, corresponding to any positive number G, however large, a number x_0 can be chosen such that $f(x)>G$ when $x\geqslant x_0$.*

We can give *similar definitions* referring to the cases in which (*i*) $f(x)\to+\infty$ when $x\to a$, (*ii*) $f(x)\to-\infty$ when $x\to a$,

(*iii*) $f(x)\to-\infty$ when $x\to+\infty$, (*iv*) $f(x)\to+\infty$ when $x\to-\infty$,
(*v*) $f(x)\to-\infty$ when $x\to-\infty$, (*vi*) $f(x)\to l$ when $x\to-\infty$,
(*vii*) $f(x)\to l$ when $x\to 0$, (*viii*) $f(x)\to+\infty$ when $x\to 0$,
(*ix*) $f(x)\to-\infty$ when $x\to 0$.

Ex. 1. Sum up the infinite series :

$$1+\frac{6}{10}+\frac{6}{10^2}+\frac{6}{10^3}+\frac{6}{10^4}+\cdots\cdots\cdots$$

By an infinite series we mean a series in which every term is followed by another term and whose terms are formed according to a given law.

Addition is a well-defined operation always possible with regard to any particular number of values, in one and only one way. A sum is the result of addition, and cannot be arrived at till the addition is complete.

If the terms to be added are numerous to such an extent that there is no end to them, *there is no end to the addition and so there is no sum.*

Within a finite period of time we may hope, given the necessary patience and longevity, to obtain the sum of any finite number of terms of any series, even if we have no particular formula as a device to fall back upon to save time. We cannot sum an infinity of terms in any finite period of time ; the process of summation is unending and, therefore, never completed. But the sum of the 1st n terms in this case is

$$S_n=1+\frac{2}{3}\left(1-\frac{1}{10^{n-1}}\right)=1\tfrac{2}{3}-\frac{2}{3\times 10^{n-1}},$$

which remains $<1\frac{2}{3}$ for every n, *i.e.*, however large the number of terms added together be.

As n increases through positive integral values, S_n approaches nearer and nearer in value to $1\frac{2}{3}$, remaining however, less than $1\frac{2}{3}$ all the time.

$S_n-1\frac{2}{3}$ never becomes equal to zero and, therefore, S_n can never actually attain the value $1\frac{2}{3}$, however large n may be.

Obviously, the number $1\frac{2}{3}$ is the only number to which S_n is approaching nearer and nearer, as n increases, and the difference $1\frac{2}{3}-S_n$ will ultimately become and remain less than

any pre-assigned positive number ϵ, however small, for all values of n greater than a certain number m (to be determined.)

Therefore, S_n has the limit $1\frac{2}{3}$. This limit is called the sum of the infinite series. It should be observed that this usage of the word 'sum' is different from its ordinary Arithmetical usage.

We have incidentally proved that $1\cdot\dot{6}=1\frac{2}{3}$.

7. Theorems on Limits.

I. *The limit of a sum, the limit of a difference, the limit of a product and the limit of a quotient of variables which tend towards finite limits is equal to the sum, the difference, the product, and the quotient of these limits respectively provided that in the case of the quotient the limit of the denominator is not zero.*

All these cases can be easily proved. As an illustration we shall prove the case of the quotient. Let x and y be two variables having l and l_1 as their limits. If we put $x=l+\epsilon$ and $y=l_1+\epsilon_1$, it follows from the definition of a limit that ϵ and ϵ_1 tend to zero.

$$\frac{x}{y}-\frac{l}{l_1}=\frac{l+\epsilon}{l_1+\epsilon_1}-\frac{l}{l_1}=\frac{\epsilon l_1-l\epsilon_1}{l_1(l_1+\epsilon_1)}.$$

As l_1 is not zero, $\dfrac{\epsilon l_1-l\epsilon_1}{l_1(l_1-\epsilon_1)}$ can be made less than any preassigned positive number, however small. Hence the limit of $\dfrac{x}{y}$ is $\dfrac{l}{l_1}$.

It may be observed that this theorem is true for a finite number of variables but is not necessarily true when the number is unlimited.

II. *A variable which lies constantly between two other variables which have the same limit (finite or infinite) tends also to the same limit.*

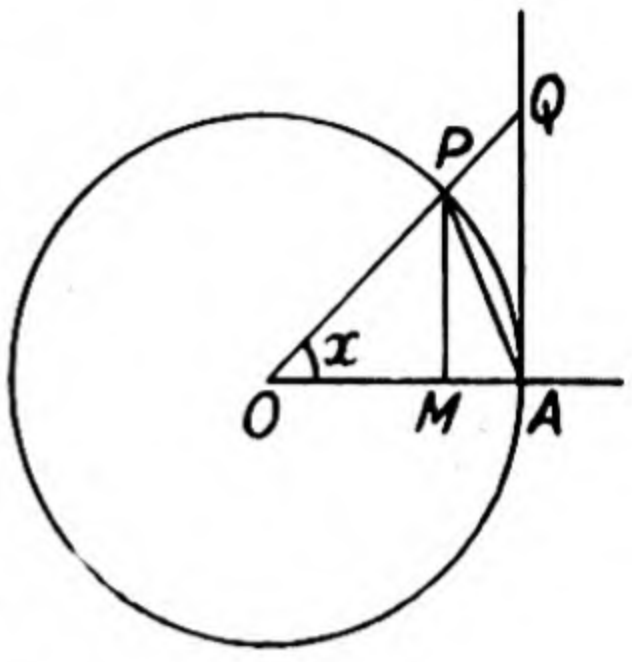
III. *If two variables are constantly equal and if one of these tends to a limit (finite or infinite), the other variable also tends to the same limit.*

Ex. 1. Prove that

$$\operatorname*{Lt}_{x\to 0}\frac{\sin x}{x}=1,$$

x being expressed in radians.

Draw a circle of *unite radiu* whose centre is O.



Take two points A and P on the circle such that $\angle AOP = x$. Draw AQ $\perp$ to OA to meet OP produced in Q.

Draw PM $\perp$ OA.

Now area of $\triangle OMP <$ area of sector OAP $<$ area of $\triangle$ OAQ (α)

or area of $\triangle$ OAP $<$ area of sector OAP $<$ area of $\triangle$ OAQ (β)

Hence, for $0 < x < \frac{\pi}{2}$,

$\frac{1}{2} \sin x \cos x < \frac{1}{2} x < \frac{1}{2} \tan x$. from ($\alpha$)

or $\frac{1}{2} \sin x < \frac{1}{2} x < \frac{1}{2} \tan x$. from ($\beta$)

i.e., $$\cos x < \frac{x}{\sin x} < \frac{1}{\cos x},$$

$$1 < \frac{x}{\sin x} < \frac{1}{\cos x}.$$

Since $\lim_{x \to 0} \cos x = \lim_{x \to 0} \frac{1}{\cos x} = 1$.

Hence either gives $\lim_{x \to +0} \frac{x}{\sin x} = 1$.

Let $x = -y$. Then

$$\lim_{x \to -0} \frac{y}{\sin y} = \lim_{x \to +0} \frac{x}{\sin x} = 1.$$

Hence $\lim_{x \to 0} \frac{x}{\sin x} = 1$ and $\therefore \lim_{x \to 0} \frac{\sin x}{x} = 1$.

Ex. 2. Show that

$$\lim_{h \to 0} \frac{\cos (x+h) - \cos x}{h} = -\sin x.$$

For $$\frac{\cos (x+h) - \cos x}{h} = -\frac{2 \sin \frac{h}{2} \sin \left(x + \frac{h}{2} \right)}{h}.$$

Also $\lim_{h \to 0} \sin \left(x + \frac{h}{2} \right) = \sin x$,

and $\lim_{h\to 0} \dfrac{\sin \frac{h}{2}}{\frac{h}{2}} = 1.$

$\therefore \lim_{h\to 0} \dfrac{\cos(x+h)-\cos x}{h} = -\sin x.$

Ex. 3. Prove that $\lim_{x\to 0} \dfrac{\tan x}{x} = 1.$

$\dfrac{\tan x}{x} = \dfrac{\sin x}{x} \cdot \dfrac{1}{\cos x}$. Proceeding to the limits, we get the result.

We assume the following results :—

(*a*) If $\lim_{x\to a} f(x) = u > 0$, then

$$\lim_{x\to a} [f(x)]^{\lambda} = u^{\lambda} = [\lim_{x\to a} f(x)]^{\lambda}.$$

(*b*) $\lim_{x\to\lambda} a^{x} = a^{\lambda} = a^{\lim x}$.

(*c*) If $\lim_{x\to a} f(x) = u > 0$, then

$$\lim_{x\to a} \log f(x) = \log u = \log \lim_{x\to a} f(x).$$

8. Passing to the limits.

When there is a relation between variables which is true for an infinity of values, then it follows from the foregoing principles that we can replace in that relation the variables by their limits. This is called *passing to the limits.*

9. To prove that $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{n} = e$, when n is a positive integer.

Expanding by the Binomial Theorem the expression becomes a sum of $n+1$ terms :

$$1 + n\frac{1}{n} + \frac{n(n-1)}{1.2}\frac{1}{n^2} + \ldots\ldots + \frac{n(n-1)\ldots(n-r+1)}{1.2.3\ldots r}\frac{1}{n^r} + \ldots + \frac{1}{n^n}$$

$$=1+1+\tfrac{1}{2}\left(1-\frac{1}{n}\right)+\cdots\cdots+\frac{1}{1.2\ldots r}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$
$$\ldots\ldots\left(1-\frac{r-1}{n}\right)+\ldots\ldots\ldots$$

Now every term as well as the number of terms increases with n so that the expression is an increasing function.

Also the expression is bounded, for the $(r+1)$th term is less than

$$\frac{1}{1.2\ldots r}<\frac{1}{2^{r-1}}$$

and the whole sum therefore is less than

$$1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^n}<3, \text{ for all } n.$$

Hence by Art. 2 the expression has a limit; this limit is usually denoted by e. It is obvious that e lies between 2 and 3.

(*i*) To deduce that $\underset{x\to\infty}{\text{Lt}}\left(1+\frac{1}{x}\right)^x=e$, where x tends to ∞ through positive real values.

Each x will be between two integers n and $n+1$; or $n\leqslant x<n+1$.

Then $1+\frac{1}{n}>1+\frac{1}{x}>1+\frac{1}{n+1}$.

$$\therefore\left(1+\frac{1}{n}\right)^{n+1}>\left(1+\frac{1}{x}\right)^x>\left(1+\frac{1}{n+1}\right)^n$$

$$i.e.,\ \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n}\right)^n>\left(1+\frac{1}{x}\right)^x>\left(1+\frac{1}{n+1}\right)^{n+1}\times\frac{1}{1+\frac{1}{n+1}}$$

Proceeding to the limits, we get the result.

(*ii*) To show that $\underset{x\to-\infty}{\text{Lt}}\left(1+\frac{1}{x}\right)^x=e$.

Putting $x=-y$, we get

$$\left(1+\frac{1}{x}\right)^{x}=\left(1-\frac{1}{y}\right)^{-y}=\left(1+\frac{1}{y-1}\right)^{y}$$

$$=\left(1+\frac{1}{z}\right)\left(1+\frac{1}{z}\right)^{z}, \text{ where } y-1=z.$$

Now when $x\to-\infty$, $y\to+\infty$ and $\therefore$ $z\to+\infty$.

Thus the result follows at once from (i)

(*iii*) To prove that $\lim_{z\to 0}(1+z)^{\frac{1}{z}}=e$.

Put $x=\frac{1}{z}$ in (i) and (ii), and we get the result.

(*iv*) To prove that $\lim_{z\to 0}(1+\mu z)^{\frac{1}{z}}=e^{\mu}$

$$(1+\mu z)^{\frac{1}{z}}=\left((1+\mu z)^{\frac{1}{\mu z}}\right)^{\mu}=\left((1+u)^{\frac{1}{u}}\right)^{\mu}, \text{ where } u=\mu z.$$

$$\therefore \lim_{z\to 0}(1+\mu z)^{\frac{1}{z}}=\lim_{u\to 0}\left((1+u)^{\frac{1}{u}}\right)^{\mu}$$

$$=\left(\lim_{u\to 0}(1+u)^{\frac{1}{u}}\right)^{\mu}=e^{\mu}.$$

10. (*i*) To prove that $\lim_{x\to 0}\frac{\log(1+x)}{x}=1$.

$$\lim_{x\to 0}\frac{\log(1+x)}{x}=\lim_{x\to 0}\log(1+x)^{\frac{1}{x}}$$

$$=\log\lim_{x\to 0}(1+x)^{\frac{1}{x}}=\log e=1.$$

(*ii*) To prove that $\lim_{x\to 0}\frac{a^{x}-1}{x}=\log a$.

Put $a^x-1=u$. Then $\underset{x\to 0}{\text{Lt}}\, u=0$.

Now $\underset{x\to 0}{\text{Lt}}\, \dfrac{a^x-1}{x}=\underset{u\to 0}{\text{Lt}}\, \dfrac{u\log a}{\log(1+u)}=\dfrac{\log a}{\underset{u\to 0}{\text{Lt}}\, \dfrac{\log(1+u)}{u}}$

$=\log a.$

As a **corollary** from above by taking $a=e$ we obtain

$$\underset{h\to 0}{\text{Lt}}\, \frac{e^h-1}{h}=1.$$

Note. Putting $h=\log\left(1+\dfrac{z}{n}\right)$ in the above equality, we have, since $e^h=1+\dfrac{z}{n}$.

$$\underset{n\to\infty}{\text{Lt}}\, \frac{z}{n}\Big/\log\left(1+\frac{z}{n}\right)=1$$

i.e., $\underset{n\to\infty}{\text{Lt}}\, n\log\left(1+\dfrac{z}{n}\right)=z$

or $\underset{n\to\infty}{\text{Lt}}\left(1+\dfrac{z}{n}\right)^n=e^z$

(*iii*) To prove that $\underset{x\to a}{\text{Lt}}\, \dfrac{x^\lambda-a^\lambda}{x-a}=\lambda a^{\lambda-1}$

Put $x=a+az$. Then

$$\frac{x^\lambda-a^\lambda}{x-a}=a^{\lambda-1}\frac{(1+z)^\lambda-1}{z}$$

Now when $x\to a$, $z\to 0$.

$$\therefore \underset{x\to a}{\text{Lt}}\, \frac{x^\lambda-a^\lambda}{x-a}=a^{\lambda-1}\underset{z\to 0}{\text{Lt}}\, \frac{(1+z)^\lambda-1}{z}$$

Put $u=(1+z)^\lambda-1$, $\lambda\neq 0$. Then $\underset{z\to 0}{\text{Lt}}\, u=0$

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Now $\underset{u\to 0}{\text{Lt}}\ \dfrac{\log(1+u)}{u}=1.$

$\therefore\ \underset{u\to 0}{\text{Lt}}\ \dfrac{\log(1+z)^{\lambda}}{(1+x)^{\lambda}-1}=1,$ or $\underset{z\to 0}{\text{Lt}}\ \dfrac{(1+z)^{\lambda}-1}{\log(1+z)}=\lambda.$

or $\underset{z\to 0}{\text{Lt}}\ \dfrac{(1+z)^{\lambda}-1}{z}\cdot\dfrac{z}{\log(1+z)}=\lambda.$

$\therefore\ \underset{z\to 0}{\text{Lt}}\ \dfrac{(1+z)^{\lambda}-1}{z}=1.$

$\therefore\ \underset{z\to 0}{\text{Lt}}\ \dfrac{z}{\log(1+z)}=1.$

Hence $\underset{x\to a}{\text{Lt}}\ \dfrac{x^{\lambda}-a^{\lambda}}{x-a}=\lambda a^{\lambda-1},$

Cor. $\underset{x\to 1}{\text{Lt}}\ \dfrac{x^{n}-1}{x-1}=n.$

EXAMPLES II

1. Evaluate $\underset{x\to 2}{\text{Lt}}\ \dfrac{x^2-4}{x-2}.$

2. $\underset{x\to 1}{\text{Lt}}\ \dfrac{x^5-1}{x^3-1}.$

3. Evaluate $\underset{x\to 0}{\text{Lt}}\ \dfrac{x^5-a^5}{x^2-a^2}.$

4. $\underset{x\to a}{\text{Lt}}\ \dfrac{x^m-a^n}{x^m-a^m}$ m, n being positive integers.

5. $\underset{x\to\infty}{\text{Lt}}\ \dfrac{1-x}{x^4}.$

6. $\underset{x\to\infty}{\text{Lt}}\ \dfrac{ax^n+bx^{n-1}+\ldots\ldots+r}{a_1x^m+b_1x^{m-1}+\ldots+r_1}$

7. $\underset{x\to 0}{\text{Lt}}\ \dfrac{a^x-1}{b^x-1}$

8. (*i*) $\underset{h\to 0}{\text{Lt}}\ \dfrac{\sin(x+h)-\sin x}{h}.$

(*ii*) $\underset{h\to 0}{\text{Lt}}\ \dfrac{\sec(x+h)-\sec x}{h}.$

9. Show that $\underset{x\to 0}{\text{Lt}}\ \dfrac{\tan x-\sin x}{x^2}=\dfrac{1}{2}.$

10. Find the limits of the following when $x\to 0$

(*i*) $\dfrac{\sin 2x}{\sin 5x}.$ (*ii*) $\left(\dfrac{\sin ax}{\sin bx}\right)^{k}.$ (*iii*) $\dfrac{1-\cos 4x}{x^2}.$

11. Show that $\lim_{x\to 0}\frac{\sin^{-1}x}{x}=1$ and $\lim_{x\to 0}\frac{\operatorname{cosec} x-\cot x}{x}=\frac{1}{2}$.

12. $\lim_{\theta\to 0}\frac{1-\cos\theta}{\theta^2}$.

13. $\lim_{x\to 0}\frac{\sin 3x\cos 2x}{\sin 2x}$.

14. $\lim_{x\to 0}\frac{1-\cos x}{\sin^2 x}$.

15. Prove that $\lim_{x\to\infty} e^{-x}\cos x=0$.

[Use $-e^{-x}\leqslant e^{-x}\cos x\leqslant e^{-x}$.]

16. The opposite angles of a cyclic quadrilateral are supplementary. Obtain a limit of this theorem when one of the quadrilateral moves round the circle towards coincidence with one of the adjacent vertices.

17. Show that if $\lim_{x=a} f(x)=k$, where k is not zero, then $f(x)$ keeps the same sign in the neighbourhood of a.

[*Hint.* Let $\epsilon<|k|$: From the definition of a limit $|f(x)-k|<\epsilon$ for $|x-a|<\delta$ or $k-\epsilon<f(x)<k+\epsilon$ for $|x-a|<\delta$. Now $k-\epsilon$ and $k+\epsilon$ are of the same sign as k.]

18. Show that $\frac{x^n}{\lfloor n}\to 0$ when $n\to\infty$ through positive integral values, x being any fixed number [$\lfloor n=1.2.3....r...n$, also $\lfloor n=n(n-1)(n-2).(n-r+1)...1$. Multiplying we get the values of $(\lfloor n)^2$. Taking two corresponding factors we get,

$r(n-r+1)=n(n-r)+r\geqslant(n-r)+r$ or n, because $n-r$ is positive and the least value of r is 1. Hence we have

$$(\lfloor n)^2\geqslant n^n \qquad \therefore \lfloor n\geq n^{\frac{n}{2}}.$$

Now $$\left|\frac{x^n}{\lfloor n}\right|<\left|\frac{x^n}{n^{\frac{n}{2}}}\right| \text{ or } \left|\frac{x^n}{\lfloor n}\right|<\left(\frac{x}{n^{\frac{1}{2}}}\right)^n.$$

As x is fixed this tends to zero because for $n>|x|^2$, $\frac{x}{n^{\frac{1}{2}}}$ is a proper fraction.]

11. Infinitesimals.

A **variable** *which has zero for its limit is called an infinitesimal ;* whereas *an* **infinite** *is a variable which tends to infinity.* Infinitesimal and infinites are functions and not numbers. Constants however small are not infinitesimals. The infinitesimal of the Calculus is not the same as the infinitesimal of ordinary speech. The latter commonly means an exceedingly small number or quantity and is usually understood to have a fixed value. In Calculus no number however small can be an infinitesimal unless it tends to zero.

Two related infinitesimals are said to be of the same order if the limits of their ratio is finite and different from zero. Usually one infinitesimal, say α, is elected as the *principal infinitesimal* and the infinitesimals are classified with reference to it. For example, an infinitesimal β is said to be of the first order if it is of the same order as α, that is to say, if limit of $\frac{\beta}{\alpha}$ is infinite and different from zero.

An infinitesimal β is said to be of order n if it is of the same order as α^n.

Let α and β be two infinitesimals. If $\operatorname{Lt} \frac{\beta}{\alpha} = 0$, then β is said to be an infinitesimal of higher order than α and α an infinitesimal of lower order than β.

Ex. 1. It is known that $\operatorname{Lt}_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$.

Therefore if θ is taken as the principal or standard infinitesimal, then $\sin\theta$ is an infinitesimal of the first order.

Similarly $\tan\theta$ is also an infinitesimal of the first order.

Ex. 2. Show that $1-\cos\theta$ is an infinitesimal of the second order with respect to θ.

$$\operatorname{Lt}_{\theta\to 0} \frac{1-\cos\theta}{\theta^2} = \operatorname{Lt}_{\theta\to 0} \frac{2\sin^2\frac{\theta}{2}}{\theta^2} = \tfrac{1}{2} \operatorname{Lt}_{\theta\to 0} \frac{\left(\sin\frac{\theta}{2}\right)^2}{\left(\frac{\theta}{2}\right)^2} = \tfrac{1}{2},$$

which is not zero. Therefore by definition $1-\cos\theta$ is of the second order.

12. Principal part of an infinitesimal.

When an infinitesimal consists of a sum of terms which are infinitesimals of different order, then the term of the lowest order is called the **principal part** of the infinitesimal.

Equivalent Infinitesimals. Two infinitesimals α and β are said to be equivalent if $\text{Lt } \frac{\beta}{\alpha} = 1$.

It follows that an infinitesimal and its principal part are equivalent.

***13. Theorems on Infinitesimals.**

(1) *The finite limit of the ratio of two infinitesimals α and β is not changed when either is replaced by any other infinitesimal α_1 or β_1 provided that* $Lt \frac{\alpha}{\alpha_1} = 1$ *and* $Lt \frac{\beta}{\beta_1} = 1$.

For $\frac{\alpha}{\beta_1} = \frac{\alpha}{\beta} \times \frac{\beta}{\beta_1}$, $\therefore \text{Lt } \frac{\alpha}{\beta_1} = \text{Lt } \frac{\alpha}{\beta} \times \text{Lt } \frac{\beta}{\beta_1} = \text{Lt } \frac{\alpha}{\beta}$

$\frac{\alpha_1}{\beta_1} = \frac{\alpha_1}{\alpha} \times \frac{\alpha}{\beta} \times \frac{\beta}{\beta_1}$, $\therefore \text{Lt} \frac{\alpha_1}{\beta_1} = \text{Lt } \frac{\alpha_1}{\alpha} \times \text{Lt } \frac{\alpha}{\beta} \times \text{Lt } \frac{\beta}{\beta_1}$

$= \text{Lt } \frac{\alpha}{\beta}$.

(2) *The limit of the quotient of two infinitesimals is not changed when either is replaced by its principal part.* Proof is obvious.

14. From the theorems on infinitesimals the following facts easily follow :

(*i*) In any sum or quotient of infinitesimals we can replace each term by its principal part, that is to say, we can neglect all infinitesimals of higher order, without affecting the limits in any way.

(*ii*) In any equation involving infinitesimals we can neglect all except those of the lowest order provided that at the same time we replace these by their principal parts.

It is wrong to say that in any equation between infinitesimals of different orders, none but those of the lowest order need be retained. This leads us to wrong results ; in fact from equations we are led to approximations. For example consider the equation

$$\theta = \tan\theta + (\theta - \tan\theta).$$

When θ tends to zero, it can be regarded as the principal infinitesimal ; θ and $\tan\theta$ are of the same order, whereas $\theta-\tan\theta$ is of the third order. If we neglect this infinitesimal of the higher order we get

$$\theta=\tan\theta$$

a relation, now, which is only approximately correct.

15. Continuous functions.

A function $f(x)$ is said to be continuous for $x=a$ if it tends to a limit as x tends to a from either side and each of these limits is equal to $f(a)$. In other words, given ϵ, positive and however small, a number δ (depending on ϵ) can be found such that

$$|f(x)-f(a)|<\epsilon \text{ for } |x-a|<\delta.$$

It follows that $f(a)-\epsilon<f(x)<f(a)+\epsilon$ whenever $a-\delta<x<a+\delta$.

If the limit of $f(x)$ when x tends to a through values greater than a is $f(a)$ whereas limit of $f(x)$ when x tends to a through values less than a is either different from $f(a)$ or does not exist, the function is said to be *continuous to the right of a.* Symbolically, if

$$\lim_{x\to a+0} f(x)=f(a) \text{ but } \lim_{x\to a-0} f(x)\neq f(a)$$

or does not exist, then $f(x)$ is said to be continuous to the right of a. Similarly if

$$\lim_{x\to a-0} f(x)=f(a) \text{ but } \lim_{x\to a+0} f(x)\neq f(a)$$

or does not exist, then $f(x)$ is said to be *continuous to the left of "a."*

$f(x)$ is said to be continuous in the interval (a, b) if it is continuous for every value of x lying between a and b and to the right of a and to the left of b.

From the corresponding theorems on limits it follows that

(*i*) *the algebraic sum as well as the product of a finite number of functions of x continuous at $x=a$ is a function continuous at $x=a$.*

(*ii*) *the quotient of two functions continuous at $x=a$ is a function continuous at $x=a$, provided that the denominator does not vanish for $x=a$.*

Ex. 1. A polynomial in x is continuous for all values of x.

If $x \to a$, $x^n \to a^n$, where n is a positive integer,

$$\therefore \quad \operatorname*{Lt}_{x \to a \pm 0} x^n = a^n. \quad \therefore \quad x^n \text{ is continuous for all values of } x.$$

Hence the polynomial $a_0x^n + a_1x^{n-1} + \ldots \ldots + a_{n-1}x + a_n$ is continuous for all values of x.

Ex. 2. The rational algebraic functions of x are continuous for all values of x for which they are defined.

Let $f(x) = \dfrac{a_0x^n + a_1x^{n-1} + \ldots \ldots + a_{n-1}x + a_n}{b_0x^m + b_1x^{m-1} + \ldots + b_{m-1}x + b_m} = \dfrac{P_1}{P_2}$, then

$f(x)$ is defined for all values of x except those which make P_2 vanish.

Now P_1 and P_2 are everywhere continuous.

$\therefore \dfrac{P_1}{P_2}$ is continuous for all values of x except at the roots of $P_2 = 0$.

Ex. 3. $\operatorname*{Lt}_{x \to 0} \sin x = 0$ and $\operatorname*{Lt}_{x \to 0} \cos x = 1$.

For however small $\epsilon > 0$ is taken, there exists an arc $\delta > 0$ of a circle of unit radius such that

$$|\sin x| < \epsilon \text{ and } 1 - \cos x < \epsilon \text{ for } |x| < \delta.$$

Ex. 4. $\operatorname*{Lt}_{x \to a} \sin x = \sin a$ and $\operatorname*{Lt}_{x \to a} \cos x = \cos a$.

For, let $x = a + \epsilon$ where ϵ, is infinitesimal. Then

$$\sin x = \sin(a + \epsilon) = \sin a \cos \epsilon + \cos a \sin \epsilon.$$

$$\therefore \operatorname*{Lt}_{x \to a} \sin x = \operatorname{Lt} \sin a \cos \epsilon + \operatorname{Lt} \cos a \sin \epsilon = \sin a.$$

Similarly for $\cos x$.

Hence it follows that $\sin x$ and $\cos x$ are continuous for all values of x.

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As a consequence of this result we have the following *theorem* :—

The circular functions are continuous at every point for which they are defined.

For $\tan x = \dfrac{\sin x}{\cos x}$, $\cot x = \dfrac{\cos x}{\sin x}$, $\sec x = \dfrac{1}{\cos x}$.

$$\operatorname{cosec} x = \frac{1}{\sin x}.$$

16. Discontinuous functions. *A function which is not continuous at a point is said to be discontinuous at that point.*

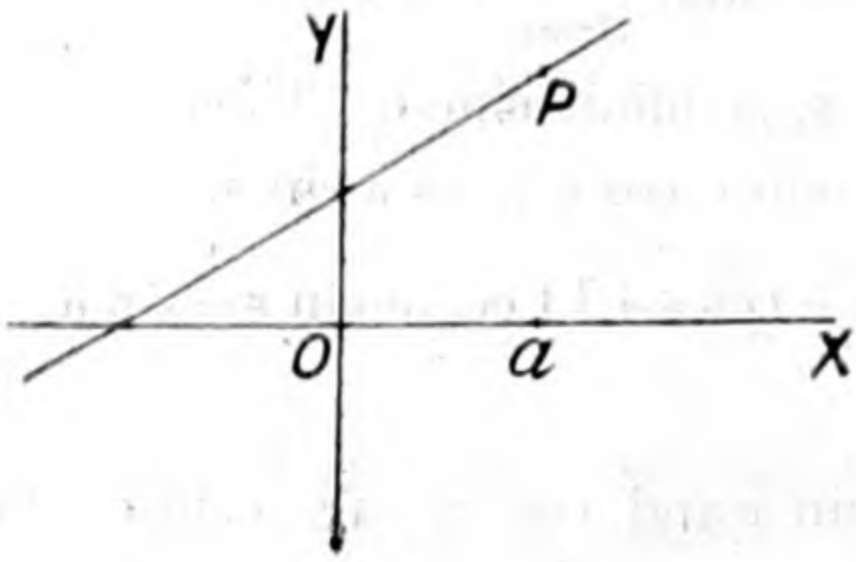
Discontinuities can arise in many ways.

The simplest type of discontinuity occurs at those points at which the function has a definite limit as x tends to the point from the right and a definite limit as x tends to the point from the left, but these two limits are different; a discontinuity also arises when these two limits are equal but different from the actual value. Discontinuities can arise in other ways also.

Ex. 1. Consider the function

$$f(x) = \frac{x^2-a^2}{x-a} = \frac{(x+a)(x-a)}{x-a}.$$

When x is not equal to a, then $f(x)=x+a$ which is continuous for all values of x. But for $x=a$, $f(x)$ is not defined. However $\underset{x=a}{\text{Lt}}\, f(x)=2a$. The function $f(x)=\dfrac{x^2-a^2}{x-a}$ is continuous for all values of x except for $x=a$ for which the actual value of the function is not defined. The graph of the function can now be easily drawn.



It may be noticed that the graph is not continuous, there being a break at the point P corresponding to $x=a$.

However the funtion $f(x)$ defined as follows is continuous for all values of x :

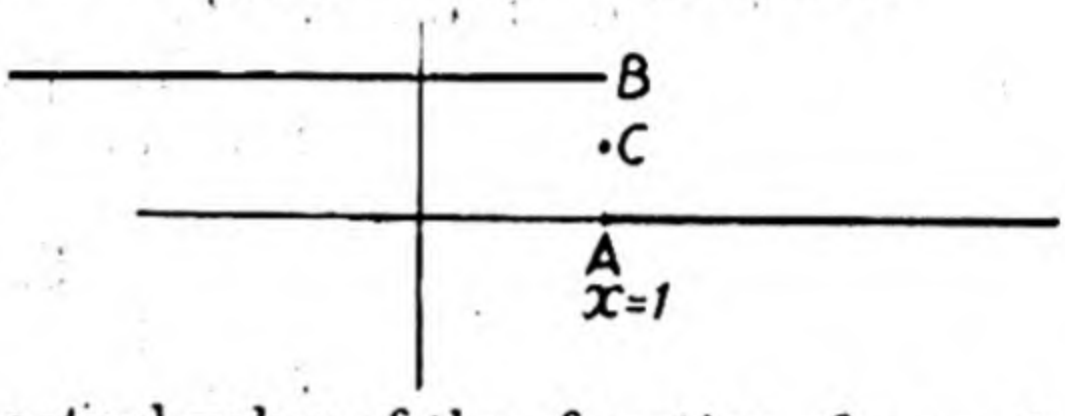
$$f(x)=\frac{x^2-a^2}{x-a} \text{ for all values of } x \text{ except } x=a,$$

and $f(a)=2a$.

The limiting value at $x=a$ being $2a$ now equals the actual value which is by definition $2a$. The graph of this function is the same as before with the difference that now there is no break at P.

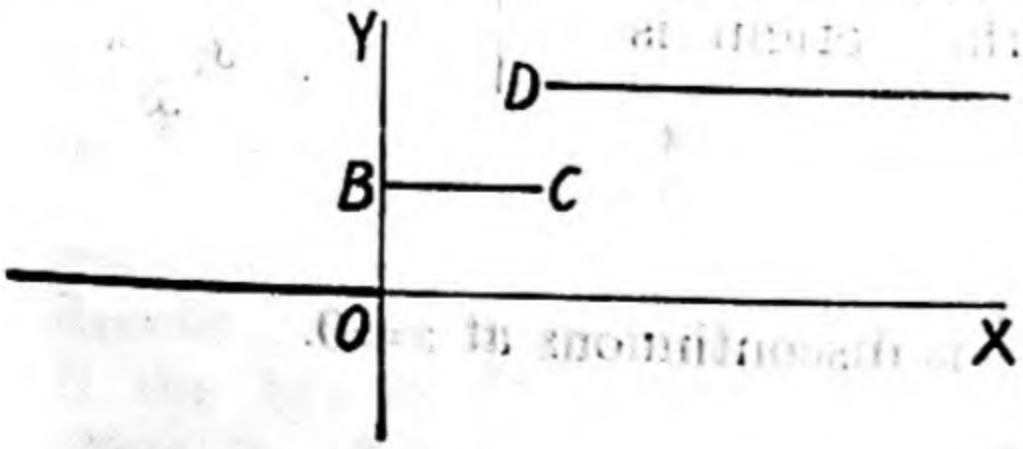
Ex. 2. Consider the function $f(x)$ defined as follows :

$f(x)=0$ for $x>1$, $f(x)=1$ for $x<1$ and $f(1)=\frac{1}{2}$. When x tends to 1 from the right the limit of $f(x)$ is 0 and as x tends to 1 from the left the limit is 1 and the actual value of the function for $x=1$ is $\frac{1}{2}$. The graph of the function is as shown in the figure. The end points A and B of the thick lines at $x=1$ are not a part of the graph. The point C which is mid-way between A and B corresponds to $x=1$.



Ex. 3. A function $f(x)$ is defined as follows : It is equal to 0 for $x<0$; equal to 1 for $0<x<1$ and equal to 2 for $x>1$; show that it has two points of discontinuity ; also draw the graph of the function.

As x tends to zero through negative values the limit of the function is 0, as x tends to zero through positive values the limit of $f(x)$ is 1. These limits being different the function is discontinuous at $x=0$. Also as x tends to 1 from the right, the limit of $f(x)$ is 2, but as x tends to 1 from the left the limit of $f(x)$ is 1 ; as these two limits are different, the function is discontinuous at $x=1$.

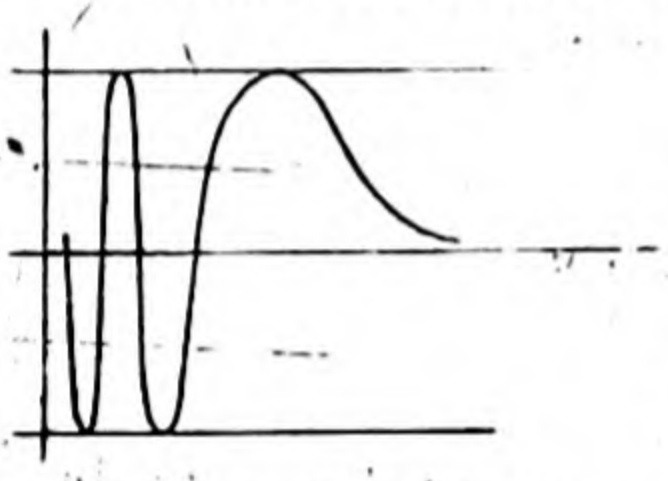


The end points B and D of the thick lines are not parts of the graph.

Ex. 4. Draw the graph of $\sin\frac{1}{x}$.

If $y=\sin\frac{1}{x}$, then $y=0$ when $x=\frac{1}{n\pi}$ where n is any integer ; $y=1$ when $x=\frac{1}{2n\pi+\frac{\pi}{2}}$ and $y=-1$ when $x=\frac{1}{\left(2n\pi-\frac{\pi}{2}\right)}$

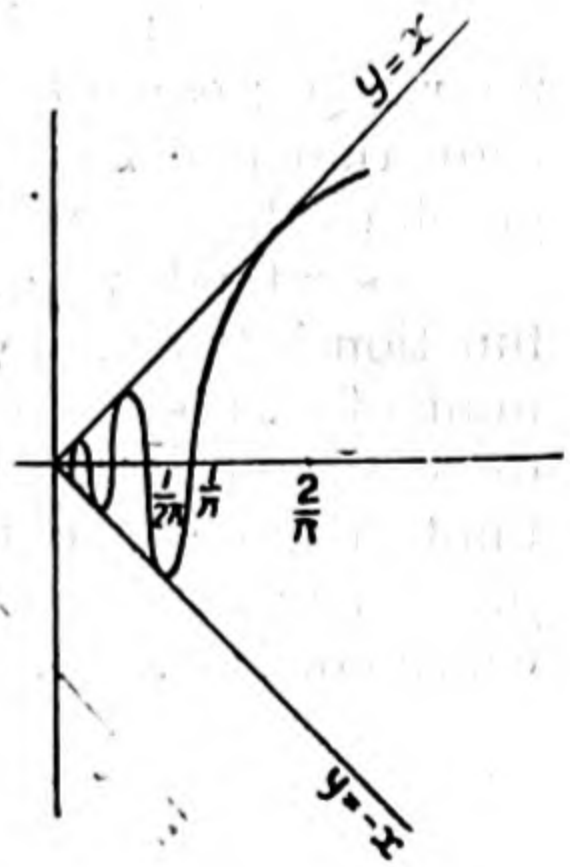
The function lies between -1 and 1 and the graph of the function therefore lies between the two lines $y=-1$ and $y=1$. It oscillates up and down and as x approaches 0 it oscillates more and more rapidly and the limit of the function does not exist, so that the function is discontinuous at $x=0$. Also the function is not defined for $x=0$.



The part of the graph when x is negative can be easily drawn.

Ex. 5. Draw the graph of $x\sin\frac{1}{x}$.

It may be observed that as $x\sin\frac{1}{x}$ always lies between 1 and -1, the function lies between x and $-x$ and the graph therefore is comprised between the lines $y=x$ and $y=-x$. As x tends to 0 the function also tends to 0 and the actual value of the function at $x=0$ is not defined. If the actual value for $x=0$ were defined to be 0, the function would become continuous at $x=0$.



The function oscillates up and down an infinite number of times, the magnitude of these oscillations becoming smaller and smaller as the origin is approached.

Ex. 6. Show that

$$f(x)=\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$ is discontinuous at $x=0$.

Now $e^{\frac{1}{x}} \to +\infty$ when $x \to +0$ (*i. e.*, through +ve values)

and $e^{\frac{1}{x}} \to 0$ when $x \to -0$ (*i. e*, through −ve values)

$$\therefore \quad \lim_{x \to +0} (fx) = \lim_{x \to +0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = \lim_{x \to +0} \frac{1 - \frac{1}{e^{\frac{1}{x}}}}{1 + \frac{1}{e^{\frac{1}{x}}}} = 1$$

$$\text{and} \quad \lim_{x \to -0} f(x) = \lim_{x \to -0} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1} = -1.$$

Also for $x=0$ the function is not defined.

It may be added that even if the function is defined for $x=0$ and assigned any value, the function could not be made continuous at $x=0$.

17. Properties of continuous functions.

In these theorems the intervals considered are all closed.

Theorem I.

If $f(x)$ is continuous in the interval (a, b) and ϵ is a positive number, given in advance, then we can divide the interval (a, b) into a finite number of sub-intervals in each of which the oscillation of $f(x)$ is less than ϵ.

Suppose that it is impossible to so divide (a, b) into sub-intervals such that the oscillation of $f(x) < \epsilon$ in each of these sub-ntervals. Divide the interval (a, b) into two equal parts (a, c_1) and (c_1, b), where $c_1 = \frac{b+a}{2}$. Then the impossibility must exist in at least one of these sub-intervals. Let (a_1, b_1) denote the sub-interval in which the impossibility exist; and if the impossibility exists in both the sub-intervals so that there is choice, then we shall choose the left hand sub-

interval. It is clear that a_1 is either equal to $\frac{b+a}{2}$ or a; and b_1 is either equal to $\frac{b+a}{2}$ or b. Again subdivide the interval (a_1, b_1) in which the impossibility exists into two equal parts with the help of the point $\frac{b_1+a_1}{2}$ and let (a_2, b_2) denote the half in which the impossibility exists and, as before, in case of choice we shall select the half to the left. So that either $a=a_1$ or $\frac{b_1+a_1}{2}$, and $b_2=\frac{b_1+a_1}{2}$ or b_1. Similarly subdivide the interval (a_2, b_2) and carry on the process in this way. Thus we form two sequences of a numbers $a_1, a_2 \ldots\ldots a_n \ldots\ldots$, and $b_1, b_2 \ldots\ldots b_n \ldots\ldots$ The sequence $a_1, a_2, \ldots\ldots a_n \ldots\ldots$ is monotonically increasing and bounded above because each of them is $< b$ whereas the other sequence is monotonically decreasing and bounded below because each of them is greater than a. Therefore both the sequences tend to finite limits. Also

$$b_1-a_1=\frac{b-a}{2}, \quad b_2-a_2=\frac{b_1-a_1}{2}=\frac{b-a}{2^2} \text{ and so on;}$$

$$b_n-a_n=\frac{b-a}{2^n}.$$

Therefore Lt $(b_n-a_n)=0$ and therefore Lt $b_n=$Lt $a_n=c$, say. It is obvious that the point c is contained in the interval (a_n, b_n) which can be made as small as we please. According to our supposition of impossibility, the oscillation of the function in (a_n, b_n) is greater than ϵ and consequently proceeding to the limit we see that the oscillation of $f(x)$ at the point c is greater than ϵ. Therefore $f(x)$ is not continuous at c, which lies in (a, b). Hence we are led to a contradiction. Hence the theorem must be true.

Cor. In this case it follows that if x' and x'' be points in any one of the sub-intervals, then $|f(x')-f(x'')| < \epsilon$, because the oscillation of $f(x)$ in any sub-interval is greater than or equal to $|f(x')-f(x'')|$.

***Theorem II.**

If $f(x)$ is continuous in (a, b); then given a positive

number ϵ, however small, a number δ (depending on ϵ) can be found such that the oscillation of $f(x)$ in any sub-interval shall be less than ϵ, provided that the amplitude of the sub-interval is less than δ.

Divide the interval (a,b) into sub-intervals such that in each of these the oscillation of $f(x)$ is less than $\frac{\epsilon}{2}$. Let δ be the amplitude of the smallest of these sub-intervals.

Now re-divide the interval (a, b) into sub-intervals, each of which has an amplitude less than δ. It is evident that any sub-interval of the second method of division cannot extend over portions of more than two sub-intervals of the first method of division. Therefore the oscillation of $f(x)$ in such a sub-interval cannot exceed the sum of oscillations of $f(x)$ in two sub-intervals of the first method of division. Thus such an oscillation is less than $2\times\frac{\epsilon}{2}$, *i.e.*, less than ϵ.

***Theorem III.**

If $f(x)$ is continuous in (a, b) then $f(x)$ is bounded in that interval.

Divide (a, b) into sub-intervals such that the oscillation of $f(x)$ in each of these sub-intervals is less than ϵ; let the points of such a division be $x_1, x_2, \ldots\ldots x_{p-1}$.

Now in the first interval (a, x_1)

$$|f(x)-f(a)|<\epsilon$$

that is, $$f(a)-\epsilon<f(x)<f(a)+\epsilon$$

In particular $$f(a)-\epsilon<f(x_1)<f(a)+\epsilon \qquad \ldots(i)$$

In the second interval $$|f(x)-f(x_1)|<\epsilon$$

that is, $$f(x_1)-\epsilon<f(x)<f(x_1)+\epsilon$$

or from (i) $$f(a)-2\epsilon<f(x)<f(a)+2\epsilon$$

In particular $$f(a)-2\epsilon<f(x_2)<f(a)+2\epsilon \qquad \ldots(ii)$$

In the third interval $$|f(x)-f(x_2)|<\epsilon$$

that is $$f(x_2)-\epsilon<f(x)<f(x_2)+\epsilon$$

and from (ii) $$f(a)-3\epsilon<f(x)<f(a)+3\epsilon$$

And in particular $$f(a)-3\epsilon<f(x_3)<f(a)+3\epsilon.$$

Continuing this argument up to the last interval (x_{p-1}, b) (which is the pth interval) we get

$$f(a)-p\epsilon<f(x)<f(a)+p\epsilon.$$

Hence it follows that $f(x)$ lies between $f(a)-p\epsilon$ and $f(a)+p\epsilon$. Thus $f(x)$ is bounded above as well as below.

***Theorem IV.**

If $f(x)$ is continuous at $x=c$, and $f(c)$ is not equal to zero then a δ can be found such that $f(x)$ keeps the same sign as that of $f(c)$ in the interval $(c-\delta, c+\delta)$.

Since $f(c)$ is not zero, select ϵ such that $\epsilon < |f(c)|$; therefore $f(c)-\epsilon$ and $f(c)+\epsilon$ have the same sign, namely that of $f(c)$.

As $f(x)$ is continuous at $x=c$, therefore

$$f(x)-f(c) \quad <\epsilon \text{ for } |x-c| <\delta.$$

That is,

$$f(c)-\epsilon<f(x)<f(c)+\epsilon \text{ for } c-\delta<x<c+\delta.$$

As $f(c)-\epsilon$ and $f(c)+\epsilon$ have the same sign as that of $f(c)$, therefore it follows that $f(x)$ has the same sign as that of $f(c)$ whenever $c-\delta<x<c+\delta$, that is, in the inverval $(c-\delta, c+\delta)$.

***Theorem V.**

If $f(x)$ is continuous in (a, b), then $f(x)$ attains its bounds at least once in (a, b).

Let M be the upper bound of $f(x)$ in (a, b); then it is known that $f(x)<M$; and however small ϵ may be $f(x)>M-\epsilon$ for at least one value of x in (a, b).

Suppose that M is not attained, that is $f(x)$ is not equal to M for any value of x in (a, b).

Consider the function $M-f(x)$; it is continuous because $f(x)$ is continuous. Consequently $\dfrac{1}{M-f(x)}$ is also continuous. Now however small ϵ may be, there is a value of x for which $f(x)>M-\epsilon$ and therfore $M-f(x)<\epsilon$ and therefore

$$\frac{1}{M-f(x)} > \frac{1}{\epsilon}.$$

It follows therefore that $\dfrac{1}{M-f(x)}$ is not bounded and therefore not continuous which contradicts the hypothesis. Hence the upper bound is attained at least once in (a, b).

Similarly by considering the function $\frac{1}{f(x)-M}$, where m is the lower bound of $f(x)$ in (a, b), it can be shown that the lower bound is also attained at least once in the interval (a, b)

***Theorem VI.**

If $f(x)$ is continuous in (a, b) and if $f(a)$ and $f(b)$ have opposite signs, then $f(x)$ vanishes for at least one value of x lying in that interval.

Divide the interval (a, b) into sub-intervals such that the variation of $f(x)$ in any sub-interval is less than ϵ. If the function does not vanish at one of the points of division, the function must change its sign somewhere between two of these consecutive points of division where its absolute value must be less than ϵ, *i. e.*, $|f(x)| < \epsilon$ and therefore $\frac{1}{|f(x)|} > \frac{1}{\epsilon}$ somewhere in (a, b). This being so, it follows that $\frac{1}{f(x)}$ is not bounded in (a, b) and therefore not continuous for at least one value of x in (a, b) which is possible if $f(x)$ vanishes at least once in (a, b).

Hence $f(x)$ must vanish at least once in (a, b).

***Theorem VII.**

If $f(x)$ is continuous in (a, b), it takes in that interval at least once every value lying between $f(a)$ and $f(b)$.

Let A be a number lying between $f(a)$ and $f(b)$. Consider the function $f(x)-A$ which is continuous in (a, b). For $x=a$ and $x=b$ the function takes values $f(a)-A$ and $f(b)-A$ which are of opposite sign; therefore the continuous function $f(x)-A$ must vanish for at least one value, say c, of x lying between a and b. That is to say $f(c)-A=0$ or $f(c)=A$.

This theorem may also be stated thus: *A continuous function in (a, b) cannot pass from one value to another without passing through all the intermediate values.*

EXAMPLES III

1. Classify each of the following functions according to the indicated plan :—

Single-valued or multiple-valued; even or odd; algebraic or transcendental.

(*i*) $2xy+x+3y-1=0$ (*ii*) $y=a^x+3x$.

2. Express y explicitly in terms of x in each of the following cases. Give also the range of values of x for which y is real :—

(*i*) $x^2-y^2=4$ (*ii*) $\cos y-2x=4$.

3. For each of the following functions :—

Find the corresponding inverse functions :

Give the range of values of y for which x is real in the inverse functions :

Tell how many values of x correspond to each value of y in the inverse functions :

(*i*) $y=x^3$ (*ii*) $y=\sin x$.

4. Show that $y=\cos x$ is a transcendental function.

[If possible, let it satisfy the irreducible equation.

$y^n+R_1(x)y^{n-1}R_2(x)y^{n-2}+\ldots+R_n(x)=0$

where R's are rational functions of x.

Change x into $x+2m\pi$ (m being an integer) and remembering that y remain unchanged, we have

$$y^n+R_1(x+2m\pi)\ y^{n-1}+\ldots+R_n(x+2m\pi)=0$$

$$\therefore\ \{R_1(x+2m\pi)-R_1(x)\}\ y^{n-1}+\ldots+$$
$$\{R_n(x+2m\pi)-R_n(x)\}=0.$$

$\therefore$ y satisfies an equation of $(n-1)$th degree which is not an identity. This contradicts the assumption.

Hence $y=\cos x$ is transcendental.]

5. Show that circular functions and inverse circular functions are transcendental.

6. Are the following functions continuous in the intervals indicated ?

(*i*) $3-2x$ $\qquad -1\leqslant x<5$.

(*ii*) $\dfrac{x}{(x+1)(x-2)}$ $\qquad -2<x\leqslant 3$.

7. Examine the following functions for continuity at the origin :—

(*i*) $f(x)=\cos\dfrac{1}{x}$ when $x\neq 0$ and $f(0)=0$

(*ii*) $f(x)=x\sin\frac{1}{x}$ when $x\neq 0$ and $f(0)=3$.

8. If $f(x)=\frac{x^2+2}{x-1}$ when $x<4$ and $f(x)=\frac{\sin x}{x-2}$ where $x>4$, for what values of x is the function not defined ?

9. A function $f(x)$ is defined as follows :

$$f\ x)=0 \text{ when } x\leqslant 0 :$$
$$f(x)=1 \text{ when } 0<x\leqslant 1$$
$$\text{and } f(x)=2 \text{ when } x>1.$$

Show that the function is discontinuous at two points and draw the graph of the function.

10. Give an example of a function which has a definite value at the origin but is discontinuous there.

11. A function $f(x)$ is defined thus :—

$$f(x)=x \qquad \text{when } 0\leqslant x<1$$
$$f(x)=2 \qquad \text{when } x=1$$
$$\text{and} \quad f(x)=1+x \quad \text{when } 1<x<2.$$

Show that it is discontinuous at $x=1$ and draw its graph.

12. Show that the function

$$f(x)\begin{cases}=x^2 & \text{when } x\neq 1\\ =2 & \text{when } x=1\end{cases}$$

is discontinuous at $x=1$. Draw its graph.

[*Hint* : Here the *limit* of the function at $x=1$ is 1 but the *value* at $x=1$ is 2].

13. Show that the limit of the function $y=\frac{x^2-1}{x-1}$ at $x=1$ is 2, though y is undefined for $x=1$. Exhibit the difference between the graphs of $\frac{x^2-1}{x-1}$ and $y=x+1$.

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CHAPTER II

DERIVATIVES AND DIFFERENTIALS

1. Derivative of a function.

Let $y=f(x)$ be a function, finite and single valued in the interval (a, b) and let x be a point of this interval; let $\delta x=h$, positive or negative, be an increment (or *difference*) given to x; the corresponding increment (or *difference*) δy of the function is $f(x+h)-f(x)$ and the ratio of the increments (or *difference quotient*) is

$$\frac{\delta y}{\delta x}=\frac{f(x+h)-f(x)}{h}.$$

If this ratio tends to a limit, finite or infinite, when h tends towards zero *in any manner* whatsover, that limit is called the **derivative** of $f(x)$ at the point x. This derivative is denoted by $f'(x)$, $Df(x)$ or $D_x f(x)$ or $\frac{dy}{dx}$.

If this ratio tends to a limit when h tends to 0 through positive values, this limit is called the **derivative to the right.** In the same way if this ratio tends to a limit when h tends to 0 through negative values, this limit is called the **derivative to the left.** When these two derivatives are equal, the function possesses a unique derivative at the point x or simply a derivative, which we have defined above.

If the function possesses a derivative at every point within the interval and a derivative to the right at a and a derivative to the left at b, we say it is **derivable in the interval (a, b).**

Every function which has a finite derivative for a given value of x is continuous for that value.

Since $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=f'(x)$, it follows that if we put $\frac{f(x+h)-f(x)}{h}-f'(x)=\epsilon$, then ϵ tends to 0 as h tends to 0.

Consequently

$$f(x+h)-f(x)=h\{f'(x)+\epsilon\}$$

so that $\underset{h\to 0}{\text{Lt}}\ \{f(x+h)-f(x)\} = \underset{h\to 0}{\text{Lt}}\, h[f'(x)+\epsilon] = 0.$

Hence $f(x)$ is continuous.

Ex. 1. Find the derivative of $f(x)$ when (*i*) $f(x)=c$, a constant, (*ii*) when $f(x)=x$.

(*i*) $\underset{h\to 0}{\text{Lt}}\ \dfrac{f(x+h)-f(x)}{h} = \underset{h\to 0}{\text{Lt}}\ \dfrac{c-c}{h} = 0.$

(*ii*) $\underset{h\to 0}{\text{Lt}}\ \dfrac{(x+h)-x}{h} = \underset{h\to 0}{\text{Lt}}\ \dfrac{h}{h} = 1.$

The derivative of a constant is zero and the derivative of x *is 1.*

Ex. 2. Find the derivative of x^5 with regard to x.

$$\underset{h\to 0}{\text{Lt}}\ \frac{(x+h)^5-x^5}{h}$$

$$= \underset{n\to 0}{\text{Lt}}\ (5x^4+10x^3h+10x^2h^2+5xh^3+h^4).$$

$$= 5x^4.$$

Ex. 3. Find the derivative of $y=\sqrt{x}$.

$$\frac{dy}{dx} = \underset{h\to 0}{\text{Lt}}\ \frac{\sqrt{x+h}-\sqrt{x}}{h} = \underset{h\to 0}{\text{Lt}}\ \frac{(x+h)-x}{\sqrt{x+h}+\sqrt{x}}\cdot\frac{1}{h}$$

$$= \underset{h\to 0}{\text{Lt}}\ \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Ex. 4. Find the derivative at $x=0$ of $f(x)$ when

$$f(x)=x\,\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$ for all values of x except 0 and $f(0)=0$.

Hence $\underset{h\to 0}{\text{Lt}}\ \dfrac{f(0+h)-f(0)}{h} = \underset{h\to 0}{\text{Lt}}\ \dfrac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}$

When h tends to zero through positive values, this limit is 1 and when h tends to zero through negative values this limit

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$$\frac{dy}{dx} = \underset{h\to 0}{\text{Lt}}\ \frac{\sqrt{x+h}-\sqrt{x}}{h} = \underset{h\to 0}{\text{Lt}}\ \frac{(x+h)-x}{\sqrt{x+h}+\sqrt{x}}\cdot\frac{1}{h}$$

$$= \underset{h\to 0}{\text{Lt}}\ \frac{1}{\sqrt{x+h}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

Ex. 4. Find the derivative at $x=0$ of $f(x)$ when

$$f(x) = x\,\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$$ for all values of x except 0 and $f(0)=0$.

Hence $\underset{h\to 0}{\text{Lt}}\ \dfrac{f(0+h)-f(0)}{h} = \underset{h\to 0}{\text{Lt}}\ \dfrac{e^{\frac{1}{h}}-1}{e^{\frac{1}{h}}+1}$

When h tends to zero through positive values, this limit is 1 and when h tends to zero through negative values this limit

is -1. The derivative therefore does not exist. The right-hand derivative is 1 and the left-hand derivative is -1.

Ex. 5. Find the derivative at $x=0$ of $f(x)$ where

$f(x)=x\sin\frac{1}{x}$ for all values of x except 0 and $f(0)=0$.

$$\operatorname*{Lt}_{h\to 0}\frac{f(0+h)-f(0)}{h}=\operatorname*{Lt}_{h\to 0}\sin\frac{1}{h},$$ which does not exist.

Note. It is easy to see that the functions in examples 4 and 5 are continuous at $x=0$. These are examples of functions which are continuous for a particular value of x but the derivative there does not exist. **A continuous function does not necessarily possess a derivative.**

In fact, Weierstrass has given an exampie of a function which is continuous and yet has no derivative for any value of x. The function is

$$f(x)=\sum_{n=0}^{\infty} b^n\cos(a^n\pi x).$$

where b is a positive constant less than 1 and a is an odd integer. The proof is difficult and beyond the scope of this book.

2. Differentials.

Let $y=f(x)$ have a finite derivative $f'(x)$

i.e. $\operatorname*{Lt}_{\delta x\to 0}\frac{\delta y}{\delta x}=f'(x)$.

$\therefore \frac{\delta y}{\delta x}=f'(x)+\epsilon$, where ϵ is an infinitesimal.

i.e., $$\delta y=f'(x)\,\delta x+\epsilon\delta x. \qquad \ldots(1)$$

Now δy is an infinitesimal; the principal part of δy, *viz.* $f'(x)\delta x$ is called the **differential** of $f(x)$, *i.e.*, of y, and is denoted by dy or $df(x)$.

Thus $dy=f'(x)\,\delta x$.

In the particular case where $f(x)$ reduces to x, $f'(x)=1$ and equation (2) reduce to

$$dx=\delta x.$$

Therefore the differential of the independent variable x is identical with its increment or difference δx, which is generally arbitrary.

The formula (2) can now be replaced by

$$dy = f'(x)\, dx$$

i.e., the differential of $y = \frac{dy}{dx} \times$ differential of x ...(3)

Note that the derivative $f'(x)$ is the co-efficient of dx, in the differential of $f(x)$. It is therefore also called the **differential co-efficient** of $f(x)$.

If we divide by dx, we get

$$\frac{dy}{dx} = f'(x).$$

Therefore the derivative of a function is also equal to the quotient of the differentials of the two variables.

Differentiability.

A function $y = f(x)$ is said to be **differentiable** at the point x if it is defined and is finite in the vicinity of that point and if given an arbitrary increment δx to x the corresponding increment δy can be expressed in the form

$$\delta y = A\delta x + \epsilon \delta x,$$

where A is independent of δx and ϵ tends to zero with δx.

From the equation $\delta y = A\delta x + \epsilon \delta x$, it follows that $\frac{\delta y}{\delta x} = A + \epsilon$ and proceeding to the limit when δx tends to zero, we get,

$$f'(x) = A.$$

This shows that if $f(x)$ is differentiable, then it possesses a finite derivative. The converse also is true ; that is to say, if $f(x)$ possesses a finite derivative $f'(x)$, then it is differentiable. This follows from equation (1) above.

The symbol $\frac{dy}{dx}$ thus acquires a double meaning but (1) holds whichever meaning we choose.

3. General Rules of Differentiation.

The process by which we find the derivative or differential co-efficient of a function is called **derivation**, or more commonly **differentiation.**

In the following rules we shall assume that u, v, w...are all functions of x and that they possess finite derivatives for the values of x concerned.

1. Derivative of a sum.

Let $y=u-v+w+$...be an algebraic sum of a finite number of functions.

Let δx be an increment given to x and let δy, δu, δv, δw,... be the corresponding increments of the functions. We have

$$y+\delta y=(u+\delta u)-(v+\delta v)+(w+\delta w)+\ldots\ldots\ldots$$

or $$\delta y=\delta u-\delta v+\delta w+\ldots\ldots$$

Therefore

$$\frac{\delta y}{\delta x}=\frac{\delta u}{\delta x}-\frac{\delta v}{\delta x}+\frac{\delta w}{\delta x}+\ldots\ldots$$

Proceeding to the limit when δx tends to zero, we get

$$\frac{dy}{dx}=\frac{du}{dx}-\frac{dv}{dx}+\frac{dw}{dx}+\ldots\ldots$$

Therefore the derivative of an algebraic sum of functions is the sum of the derivatives.

Multiplying both sides by dx, we get,

$$dy=du-dv+dw+\ldots\ldots,$$

so that the rule is true about differentials also.

II. Derivative of a product.

Let $y=uv$ be a product of two functions.

We have

$$\frac{\delta y}{\delta x}=\frac{(u+\delta u)(v+\delta v)-uv}{\delta x}$$

$$=(v+\delta v)\frac{\delta u}{\delta x}+u\frac{\delta v}{\delta x}.$$

Proceeding to the limit when δx tends to 0, we get

$$\frac{dy}{dx}=\text{Lt}\frac{\delta u}{\delta x}\times\text{Lt}\,(v+\delta v)+u\text{Lt}\,\frac{\delta v}{\delta x}$$

$$=v\frac{du}{dx}+u\frac{dv}{dx}.$$

Multiplying both sides by dx, we get

$$dy=vdu+udv.$$

As a particular case if v is a constant, say c, then its derivative is zero and consequently when $y=cu$,

$$\frac{dy}{dx}=c\frac{du}{dx}\text{ and }dy=cdu.$$

This is stated by saying that *a constant factor can be taken outside the sign of differentiation.*

By applying this rule, for $y=uvw$, we get

$$\frac{dy}{dx}=vw\frac{du}{dx}+u\frac{d}{dx}(vw)$$

$$=vw\frac{du}{dx}+uw\frac{dv}{dx}+uv\frac{dw}{dx}$$

$$=uvw\left(\frac{\frac{du}{dx}}{u}+\frac{\frac{dv}{dx}}{v}+\frac{\frac{dw}{dx}}{w}\right)$$

Similarly this can be extended to any number of factors.

If $y=uvw\ldots\ldots$ to n factors, then

$$\frac{dy}{dx}=uvw\ldots\ldots\left(\frac{\frac{du}{dx}}{u}+\frac{\frac{dv}{dx}}{v}+\frac{\frac{dw}{dx}}{w}+\ldots\ldots\right)$$

If we put $u=v=w=\ldots\ldots$ we get $y=u^n$,

and $$\frac{dy}{dx}=u^n\left(n\frac{\frac{du}{dx}}{u}\right)=nu^{n-1}\frac{du}{dx}.$$

In particular, if $u=x$ then $y=x^n$ and $\frac{dy}{dx}=nx^{n-1}$.

If we multiply the equations with dx we get the corresponding formulae for differentials.

III. Derivative of a quotient.

Let $y = \frac{u}{v}$; we have $y+\delta y = \frac{u+\delta u}{v+\delta v}$,

and $\delta y = \frac{u+\delta u}{v+\delta v} - \frac{u}{v}$,

or $$\frac{\delta v}{\delta x} = \frac{\frac{u+\delta u}{v+\delta v} - \frac{u}{v}}{\delta x} = \frac{v\frac{\delta u}{\delta x} - u\frac{\delta v}{\delta x}}{v(v+\delta v)},$$

and passing to the limit when δx tends to zero, we get

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

In the particular case when u is constant, say c,

$$y = \frac{c}{v} \text{ and } \frac{dy}{dx} = -\frac{c}{v^2}\frac{dv}{dx}.$$

IV. Derivative of an inverse function.

Let $y=f(x)$ and $x=\phi(y)$ be inverse functions.

If one of these functions possesses a derivative which is not zero, the other function also possesses a derivative.

Suppose that the derivative of $\phi(y)$, *i.e.*, $\phi'(y)$ is known, and $\phi'(y)$ is not zero.

Then $\frac{\delta y}{\delta x} = \frac{1}{\frac{\delta x}{\delta y}}$; but Lt $\frac{\delta x}{\delta y} = \phi'(y)$, which is not zero.

Proceeding to the limit,

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} \text{ as } \phi'(y) \text{ or } \frac{dx}{dy} \text{ is not zero}$$

i.e., $$f'(x) = \frac{1}{\phi'(y)}.$$

Therefore the derivative of y considered as a function of x is the reciprocal of the derivative of x considered as a function of y.

V. Derivative of a function of a function.

Let $y=\phi(u)$ and $u=f(x)$ so that $y=\phi\{f(x)\}$. We say that y is a function of a function of x. We assume that $\phi(u)$ and $f(x)$ have finite derivatives $\phi'(u)$ and $f'(x)$ respectively. Let δx be an increment in x and let δu and δy be the corresponding increments in u and y. Let δx tend towards zero without actually taking the value zero; then δu and δy tend towards zero (and might even actually be zero).

As $\phi'(u)$ and $f'(x)$ exist and are finite, it follows that

$$\delta y=[\phi'(u)+\epsilon]\ \delta u,$$

and $$\delta u=[f'(x)+\epsilon']\ \delta x$$

where ϵ and ϵ' tend towards zero with δx.

Therefore $$\delta y=[\phi'(u)+\epsilon][f'(x)+\epsilon']\delta x$$

or $$\frac{\delta y}{\delta x}=[\phi'(u)+\epsilon][f'(x)+\epsilon'].$$

Proceeding to the limit, we get

$$\frac{dy}{dx}=\phi'(u)\times f'(x),$$

that is to say, $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}.$$

If we multiply both the sides by dx, we get

$$dy=\phi'(u)f'(x)dx=\phi'(u)du.$$

The form of dy is the same as if u were an independent variable.

4. We have already suggested methods by means of which the derivatives of familiar functions may be found. But since the structure of the Differential Calculus depends for its applications upon the concept of rate of change, it is

necessary to develop formulae systematically for obtaining the derivatives of standard forms. Each of these formulae must, of course, be obtained as a limit. But once we have carried out the limiting process and obtained a general rule we shall then use the rule whenever applicable.

5. Differentiation of elementary functions.

I. x^n where n is a real number.

Let $y=x^n$.

$$\text{Then } \frac{\delta y}{\delta x}=\frac{(x+\delta)^n-x^n}{\delta x}=x^n\frac{\left(1+\frac{\delta x}{x}\right)^n-1}{\delta x}$$

$$=x^n\frac{(1+u)^n-1}{xu} \qquad \left(\frac{\delta x}{x}=u\right)$$

When $\delta x\to 0$, $u\to 0$. We have proved that

$$\operatorname*{Lt}_{u\to 0}\frac{(1+u)^n-1}{u}=n.$$

$$\text{Thus } \frac{d}{dx}\left(x^n\right)=nx^{n-1} \text{ provided that } x^n \text{ is defined.}$$

Ex. 1. Show that $\frac{d}{dx}(ax+b)^n=na(ax+b)^{n-1}$.

Let $y=z^n$ where $z=ax+b$

$$\therefore \frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}=nz^{n-1}.\ a=na(ax+b)^{n-1}$$

Ex. 2. If $\phi(x)=a_0x^n+a_1x^{n-1}+\ldots+a_n$, find $\phi'(x)$.

Let $y=\phi(x)$. Then

$$\phi'(x)=\frac{dy}{dx}=\frac{d}{dx}\ (a_0x^n+ax^{n-1}+\ \ldots\ldots a^n)$$

$$=\frac{d}{dx}\left(a_0x^n+\ \frac{d}{dx}\right)a_1x^{n-1}+\ \ldots\ldots+\frac{d}{dx}(a^n)$$

$$=na_0x^{n-1}+(n-1)a_1x^{n-2}+\ldots.a^{n-1}.$$

II e^x. From definition of derivative, when $y=e^x$

$$\frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\frac{e^{x+h}-e^x}{h}=e^x\,\text{Lt}\frac{e^h-1}{x}$$

Put $e^h-1=z$ so that $h=\log(1+z)$.

As h tends to zero, z also tends to zero.

$$\underset{h=0}{\text{Lt}}\frac{e^h-1}{h}=\underset{z\to 0}{\text{Lt}}\frac{z}{\log(1+z)}$$

$$=\frac{1}{\underset{z=0}{\text{Lt}}\log(1+z)^{\frac{1}{z}}}=\frac{1}{\log e}=1.$$

Hence $$=\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{e}^x=\mathbf{e}^x.$$

Cor. If $y=a^x$, then $y=e^{x\log a}$, and applying the rule of function of a function we get,

$$\frac{dy}{dx}=e^{x\log a}\times\frac{d}{dx}(x\log a)$$

$$=e^{x\log a}\times\log a=a^x\log a.$$

III. $\log_e x$. Let $y=\log e^x$. Then $x=e^y$.

But $\dfrac{dx}{dy}=e^y=x$.

Since $\dfrac{dy}{dx}=\dfrac{1}{\frac{dx}{dy}}$ $\quad\therefore\quad \dfrac{\mathbf{dy}}{\mathbf{dx}}=\dfrac{\mathbf{1}}{\mathbf{x}}$.

Another method.

$$\frac{dy}{dx}=\underset{h=0}{\text{Lt}}\frac{\log(x+h)-\log x}{h}=\underset{h=0}{\text{Lt}}\frac{\log\left(1+\frac{h}{x}\right)}{h}$$

$$=\frac{1}{x}\underset{h=0}{\text{Lt}}\frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}}$$

$$\text{But} \lim_{h\to 0} \frac{\log\left(1+\frac{h}{x}\right)}{\frac{h}{x}} = \lim_{h\to 0} \log\left(1+\frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \log e = 1.$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{x}.$$

Ex. 1. Find the derivative of x^n where $x>0$ and n is a real number.

Let $y=x^n$.

Then $y=e^{n \log x}=e^u$ where $u=n \log x$.

Now $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$, and $\frac{dy}{du}=e^u$, $\frac{du}{dx}=\frac{n}{x}$.

$$\therefore \quad \frac{dy}{dx}=nx^{n-1}.$$

Ex. 2. Differentiate $\log \log x$.

Let $y=\log \log x$, where x is so large that $\log \log x$ is defined.

Let $u=\log x$. Then $y=\log u$.

Hence $\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{u}\cdot\frac{1}{x}=\frac{1}{x \log x}$.

By the repeated application of this result we have

$$\frac{d}{dx}(\log \log \ldots\ldots \log x)=\frac{d}{dx}(l_m x)=\frac{1}{x l_1 x l_2 x \ldots\ldots l_{m-1} x}$$

where l_m means that log is repeated m times.

Ex. 3. Differentiate $(\log x)^{1-\lambda}$, $\lambda \neq 1$.

Let $y=(\log x)^{1-\lambda}$.

Let $u=\log x$. Then $y=u^{1-\lambda}$

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1-\lambda}{u^\lambda}\cdot\frac{1}{x}=\frac{1-\lambda}{x(\log x)^\lambda}$$

Ex. 4. Find the differential co-efficient of

(*i*) e^{ax^m}. (*ii*) $\log(x+\sqrt{1+x^2})$.

(*i*) Let $y=e^u$, where $u=ax^m$.

$$\therefore \quad \frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=e^u.amx^{m-1}=amx^{m-1}e^{ax^m}$$

(*ii*) Let $y=\log u$, where $u=x+\sqrt{1+x^2}$. Then

$$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}=\frac{1}{u}\cdot\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)$$

$$=\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{x+\sqrt{1+x^2}}{\sqrt{1+x^2}}=\frac{1}{\sqrt{1+x^2}}.$$

EXAMPLES IV

Find the derivatives of

1. $\frac{1}{x^3}$. 2. $x+\sqrt{1+x^2}$. 3. $\frac{1}{\sqrt{1+x^2}+x}$.

4. $x^m(1-x)^n$. 5. $\sqrt{a^2+x^2}$. 6. $\frac{x}{\sqrt{a^2+x^2}}$.

7. $\sqrt{\frac{1-x+x^2}{1+x-x^2}}$. 8. $\log\frac{a+bx}{a-bx}$.

Find the differential co-efficient of:

9. $\log(x+\sqrt{x^2+a^2})$. 10. $\frac{\sqrt{2x+1}}{(2x+3)^2\sqrt[3]{x^2-5}}$. 11. $\frac{\log 2x}{x}$.

12. 5^{2x-5}. 13. (*i*) $a^x.x^a$. (*ii*) a^{x^n}. (*iii*) $x^ne^{ax}\log x$.

14. $\log_{10}(\sqrt{x}+1)$. 15. Differentiate $\log\sqrt{1+x+x^2}$.

IV. Circular Functions.

To find the derivatives of (*i*) $\sin x$ (*ii*) $\cos x$ (*iii*) $\tan x$ (*iv*) $\operatorname{cosec} x$ (*v*) $\sec x$ (*vi*) $\cot x$.

(*i*) Let $y=\sin(x)$. Then

$$\frac{dy}{dx}=\operatorname*{Lt}_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

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$$= \text{Lt}\ \frac{2\cos\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= \text{Lt}\ \cos\left(x+\frac{h}{2}\right)\ \text{Lt}\ \frac{\sin\frac{h}{2}}{\frac{h}{2}} = \cos x.$$

Thus $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\sin x}) = \mathbf{\cos x}.$

(*ii*) Let $y = \cos x$. Then

$$\frac{dy}{dx} = \underset{h\to 0}{\text{Lt}}\ \frac{\cos(x+h)-\cos x}{h}$$

$$= -\text{Lt}\ \frac{2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h}$$

$$= -\underset{h\to 0}{\text{Lt}}\ \sin\left(x+\frac{h}{2}\right).\ \underset{h\to 0}{\text{Lt}}\ \frac{\sin\frac{h}{2}}{\frac{h}{2}} = -\sin x$$

Thus $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\cos x}) = \mathbf{-\sin x}$

(*iii*) Let $y = \tan x$. Then

$$\frac{dy}{dx} = \underset{h\to 0}{\text{Lt}}\ \frac{\tan(x+h)-\tan x}{h}$$

$$= \text{Lt}\ \frac{\sin(x+h)\cos x - \cos(x+h)\sin x}{h\cos x\cos(x+h)}$$

$$= \text{Lt}\ \frac{\sin h}{h\cos x\ \cos(x+h)}$$

$$= \text{Lt}\ \frac{\sin h}{h}\ \text{Lt}\ \frac{1}{\cos x}.\ \text{Lt}\ \frac{1}{\cos(x+h)} = \frac{1}{\cos^2 x}.$$

Hence $\dfrac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\tan x}) = \mathbf{\sec^2 x}.$

Or

$y=\frac{\sin x}{\cos x}$. Apply (*i*), (*ii*) and Quotient Theorem.

(*iv*) Let $y=\cot x$. Then

$$\frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\frac{\cot(x+h)-\cot x}{h}$$

$$=\text{Lt}\frac{\cos(x+h)\sin x-\sin(x+h)\cos x}{h\sin x\sin(x+h)}$$

$$=-\text{Lt}\frac{\sin h}{h}\cdot\text{Lt}\frac{1}{\sin x\sin(x+h)}=-\frac{1}{\sin^2 x}$$

Hence $\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\cot x})=-\mathbf{cosec^2 x}$.

Or $y=\frac{\cos x}{\sin x}$. Apply (*i*), (*ii*) and the Quotient Theorem.

(*v*) Let $y=\text{cosec } x$. Then

$$\frac{dy}{dx}=\underset{h\to 0}{\text{Lt}}\frac{\text{cosec}(x+h)-\text{cosec } x}{h}$$

$$=-\text{Lt}\frac{\sin(x+h)-\sin x}{h\sin(x+h)\sin x}$$

$$=-\text{Lt}\frac{2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{h\sin(x+h)\sin x}$$

$$=-\text{Lt}\frac{\sin\frac{h}{2}}{\frac{h}{2}}\cdot\text{Lt}\cos\left(x+\frac{h}{2}\right)\times$$

$$\text{cosec } x\,\text{cosec}(x+h)=-\cot x\,\text{cosec } x.$$

$\therefore$ $\frac{\mathbf{d}}{\mathbf{dx}}(\mathbf{\text{cosec } x})=-\mathbf{\cot x\,\text{cosec } x}$.

Or

$y=\frac{1}{\sin x}$. **Apply (*i*) and Quotient Theorem.**

(*vi*) Let $y=\sec x$. Then

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\sec(x+h)-\sec x}{h}$$

$$=\text{Lt}\frac{\cos x-\cos(x+h)}{h\cos(x+h)\cos x}$$

$$=\text{Lt}\frac{2\sin\frac{h}{2}\sin\left(x+\frac{h}{2}\right)}{h\cos(x+h)\cos x}=\frac{\sin x}{\cos^2 x}=\tan x\sec x.$$

Hence $\dfrac{\mathbf{d}}{\mathbf{dx}}\mathbf{(sec\ x)=tan\ x\ sec\ x.}$

Or

$y=\dfrac{1}{\cos x}$. Apply (*ii*) and Quotient Theorem.

Ex. 1. Differentiate $e^{ax}\sin bx$.

Let $y=e^{ax}\sin bx$. Then

$$\frac{dy}{dx}=e^{ax}a.\sin bx+e^{ax}\cos bx.b$$

$$=e^{ax}(a\sin bx+b\cos bx).$$

V. Inverse Circular Functions.

To find the derivative of :

(*i*) $\sin^{-1}x$ (*ii*) $\cos^{-1}x$ (*iii*) $\tan^{-1}x$ (*iv*) $\cot^{-1}x$ (*v*) $\sec^{-1}x$ (*vi*) $\text{cosec}^{-1}x$.

(*i*) Let $y=\sin^{-1}x$ .....(1)

Then $x=\sin y$ .....(2)

$$\therefore\quad \frac{dx}{dy}=\cos y,\ \text{i.e.,}\ \frac{dy}{dx}=\frac{1}{\cos y}=\pm\frac{1}{\sqrt{1-x^2}}.$$

The ambigouus sign is the same as that of $\cos y$, *i.e.*, of $\cos(\sin^{-1}x)$. For the principal branch, *viz.*, for the interval $-\frac{\pi}{2}\leqslant y\leqslant\frac{\pi}{2}$, $\cos y$ is positive; hence the positive sign of the radical must be taken.

$$\therefore \quad \frac{\mathbf{d}}{\mathbf{dx}}(\sin^{-1}\mathbf{x})=\frac{1}{\sqrt{1-x^2}} \text{ for } -\frac{\pi}{2}\leqslant \sin^{-1}x \leqslant \frac{\pi}{2}.$$

The other branches of $\sin^{-1}x$ may be divided into two classes namely, $y=\sin^{-1}x+2k\pi$ and $(\pi-\sin^{-1}x)+2k\pi$. The derivative of the first class is the same as that of the principal branch and the derivative of the second class is

$$-\frac{1}{\sqrt{1-x^2}}.$$

(*ii*) Let $y=\cos^{-1}x$

Then $x=\cos y$.

$$\therefore \quad \frac{dx}{dy}=-\sin y, \text{ i.e. } \frac{dy}{dx}=-\frac{1}{\sin y}=-\frac{1}{\pm\sqrt{1-\cos^2 y}}=\mp\frac{1}{\sqrt{1-x^2}}.$$

The ambiguous sign is the same as that of $\sin y$, i.e., of $\sin(\cos^{-1}x)$.

For the principal branch, *i.e.*, for the range $0\leqslant y\leqslant\pi$, $\sin y$ is positive.

$$\therefore \quad \frac{\mathbf{d}}{\mathbf{dx}}(\cos^{-1}\mathbf{x})=-\frac{1}{\sqrt{1-\mathbf{x}^2}} \text{ for } 0\leqslant\cos^{-1}\mathbf{x}\leqslant\pi.$$

Otherwise thus :

$y=\cos^{-1}x=\frac{\pi}{2}-\sin^{-1}x$ and therefore $\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}}$

for the principal branch.

(*iii*) Let $y=\tan^{-1}x$, so that $x=\tan y$

$$\therefore \quad \frac{dx}{dy}=\sec^2 y, \text{ i.e., } \frac{dy}{dx}=\frac{1}{\sec^2 y}=\frac{1}{1+x^2}.$$

Tan y assumes all positive and negative values as y ranges over an interval of π units ; but $\sec^2 y$ is always positive for these values.

Hence for the principal range $-\frac{\pi}{2}<\tan^{-1}x<\frac{\pi}{2}$ and all other ranges of

$$\frac{\mathbf{d}}{\mathbf{dx}}(\tan^{-1}\mathbf{x})=\frac{1}{1+\mathbf{x}^2}.$$

(*iv*) Let $y=\cot^{-1}x$.

Then $x=\cot y$

$$\therefore \frac{dx}{dy}=-\operatorname{cosec}^2 y, \text{ i.e., } \frac{dy}{dx}=-\frac{1}{\operatorname{cosec}^2 y}=-\frac{1}{1+x^2}.$$

Hence for the principal range $0<\cot^{-1}x<\pi$ and all other ranges $\mathbf{\frac{d}{dx}(\cot^{-1}x)=-\frac{1}{1+x^2}}$.

(*v*) Let $y=\sec^{-1}x$ so that $x=\sec y=\frac{1}{\cos y}$.

$$\therefore \frac{dx}{dy}=\frac{\sin y}{\cos^2 y}=\tan y \sec y=\pm x\sqrt{x^2-1}.$$

The ambiguous sign is the same as that of tan y, *i.e.*, of $\tan(\sec^{-1}x)$.

For the principal range

$0\leqslant\sec^{-1}x<\frac{\pi}{2}$ if $x\geq 1$ and $-\pi\leqslant\sec^{-1}x<-\frac{\pi}{2}$

if $x\leqslant -1$, tan y is positive; hence the positive sign of the radical must be taken.

$$\therefore \mathbf{\frac{d}{dx}(\sec^{-1}x)=\frac{1}{x\sqrt{x^2-1}}} \text{ for the principal range.}$$

Or $y=\cos^{-1}\frac{1}{x}$. Apply (*ii*).

Note that $\sec^{-1}x$ is not defined for values of x between -1 and $+1$.

(*vi*) Let $y=\operatorname{cosec}^{-1}x$, then $x=\operatorname{cosec} y$

$$\therefore \frac{dx}{dy}=-\operatorname{cosec} y \cot y=-x\{\pm\sqrt{x^2-1}\}.$$

The ambiguous sign is the same as that of cot y.

For the principal range $0<y\leqslant\frac{\pi}{2}$ if $x\geq 1$ and $-\pi<y\leqslant -\frac{\pi}{2}$ if $x\leqslant -1$, cot y is positive; hence the positive sign of the radical must be taken.

$$\therefore \quad \frac{dy}{dx}=-\frac{1}{x\sqrt{x^2-1}} \text{ for the principal range.}$$

Note that $\text{cosec}^{-1}x$ is not defined for values of x between -1 and $+1$.

Ex. 2. Show that $\frac{d}{dx}[\tan^{-1}(\frac{1}{2}\tan x)]=\frac{2}{4-3\sin^2 x}$.

Let $y=\tan^{-1}(\frac{1}{2}\tan x)$. Then

$$\frac{dy}{dx}=\frac{1}{1+\frac{1}{4}\tan^2 x}\cdot\frac{1}{2}\sec^2 x=\frac{2}{4\cos^2 x+\sin^2 x}=\frac{2}{4-3\sin^2 x}.$$

EXAMPLES V

Differentiate :

1. $\sin^2 x$. **2.** $\cos mx$. **3.** $x^2\sin x$. **4.** $\frac{\tan x}{1+\tan x}$.

5. $\frac{1+\sec x}{\sec x}$. **6.** $\cos^m x$. **7.** $\sin^m x\cos^n x$. **8.** $\text{Sin } x^n$.

9. $\cos(\sin x)$. **10.** $\sin\sqrt{x}$. **11.** $\cos(\log x)$. **12.** $e^{\sin 2x}$.

13. $\log\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)$. **14.** $\log\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)$.

15. $\log\tan\frac{x}{2}$. **16.** $e^x\tan x$. **17.** $e^{ax}\sin bx$.

18. $\log\frac{a+b\tan x}{a-b\tan x}$. **19.** $\tan^{-1}\frac{1+x^2}{1-x^2}$. **20.** $\tan^{-1}\frac{a+x}{1-ax}$.

21. $\tan^{-1}(\sec x+\tan x)$. **22.** $\tan^{-1}\left(\frac{b}{a}\tan x\right)$.

23. $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$. **24.** $x^2=e^{xy}$. **25.** $x\sin y=y\tan x$.

26. $(\tan x)^k=(\tan y)^m$.

27. Show by differentiation that each of the following functions is independent of x :—

(*i*) $\tan^{-1}x+\tan^{-1}\frac{1}{x}$. (*ii*) $\sin^{-1}x+\cos^{-1}x$.

$$(iii)\quad \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}} \qquad \therefore \quad \frac{dy}{dx} = \frac{2x}{1+x^2} \cdot \sqrt{x^2-1} = \frac{2x}{x\sqrt{x^2-1}}$$

23. Differentiate the inverse circular functions of x from first principles.

Let $y=\tan^{-1}x$, so that $x=\tan y$

$y+\delta y=\tan^{-1}(x+\delta x)$ and $x+\delta x=\tan(y+\delta y)$ and $\delta x=\tan(y+\delta y)-\tan y$,

$$\therefore \quad \frac{dy}{dx} = \operatorname*{Lt}_{\delta x\to 0} \frac{\delta y}{\delta x} = \operatorname*{Lt}_{\delta y\to 0} \frac{\delta y}{\tan(y+\delta y)-\tan y}$$

$$= \operatorname*{Lt}_{\delta y\to 0} \frac{\delta y \cos\delta y \cos(y+\delta y)}{\sin \delta y}$$

$$= \cos^2 y = \frac{1}{1+x^2}.$$

Similarly the other circular functions can be treated.

Hyperbolic Functions.

To find the derivatives of (*i*) $\sinh x$ (*ii*) $\cosh x$ (*iii*) $\tanh x$ (*iv*) $\coth x$ (*v*) $\operatorname{sech} x$, (*vi*) $\operatorname{cosech} x$.

(*i*) Let $y=\sinh x=\frac{1}{2}(e^x-e^{-x})$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(e^x+e^{-x}) = \cosh x.$$

(*ii*) Let $y=\cosh x=\frac{1}{2}(e^x+e^{-x})$.

$$\therefore \quad \frac{dy}{dx} = \frac{1}{2}(e^x-e^{-x}) = \sinh x.$$

(*iii*) Let $y=\tanh x = \dfrac{\sinh x}{\cosh x}$

$$\frac{dy}{dx} = \frac{\cosh x.\cosh x - \sinh x \sinh x}{\cosh^2 x} = \operatorname{sech}^2 x.$$

27. Show by differentiation from first principles that the following functions is independent of x:—

$$\operatorname*{Lt}_{h\to 0} \frac{\tanh(x+h)-\tanh x}{h} = \operatorname*{Lt}_{h\to 0} \frac{\sinh(x+h)\cosh x - \cosh(x+h)\sinh x}{h\cosh x \cosh(x+h)}$$

$$= \operatorname*{Lt}_{h\to 0} \frac{\sinh h}{h} \cdot \frac{1}{\cosh x \cosh (x+h)} = \operatorname{sech}^2 x.$$

(*iv*) Let $y=\coth x = \dfrac{\cosh x}{\sinh x}$. Then $\dfrac{dy}{dx} = -\operatorname{cosech}^2 x$.

(*v*) Let $y=\operatorname{sech} x$. Then $\dfrac{dy}{dx} = -\tanh x \operatorname{sech} x$

(*vi*) Let $y=\operatorname{cosech} x = \dfrac{1}{\sinh x}$.

Then $\dfrac{dy}{dx} = -\coth x \operatorname{cosech} x$.

VII. Inverse Hyperbolic Functions.

To find the derivatives of (*i*) $\sinh^{-1}x$ (*ii*) $\cosh^{-1}x$ (*iii*) $\tanh^{-1}x$ (*iv*) $\coth^{-1}x$ (*v*) $\operatorname{sech}^{-1}x$ (*vi*) $\operatorname{cosech}^{-1}x$.

(*i*) Let $y=\sinh^{-1}x$ so that $x=\sinh y$

$$\therefore \frac{dx}{dy} = \cosh y, \quad i.e., \quad \frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}.$$

There is no ambiguity of sign as $\cosh y$ is positive.

$$\text{Or } \frac{dy}{dx} = \operatorname*{Lt}_{h\to 0} \frac{\sinh^{-1}(x+h) - \sinh^{-1}x}{(x+h)-x}$$

$$= \operatorname*{Lt}_{\delta y\to 0} \frac{(y+\delta y)-y}{\sinh(y+\delta)-\sinh y} = \frac{1}{\cosh y} = \frac{1}{\sqrt{1+x^2}}$$

(*ii*) Let $y=\cosh^{-1}x$, or $x=\cosh y$.

$$\therefore \frac{dx}{dy} = \sinh y = \pm\sqrt{x^2-1} \quad \text{whence} \quad \frac{dy}{dx} = \pm\frac{1}{\sqrt{(x^2-1)}}.$$

For any given value of $x>1$, there are two values of y and for these values $\dfrac{dy}{dx}$ has opposite signs.

(*iii*) Let $y=\tanh^{-1}x$, or $x=\tanh y$, $|x|<1$.

$$\therefore \quad \frac{dy}{dx} = \operatorname{sech}^2 y = 1-x^2$$

$$\therefore \quad \frac{dy}{dx} = \frac{1}{1-x^2}.$$

(*iv*) Let $y=\coth^{-1} x$, or $x=\coth y$, $|x|>1$

$$\therefore \quad \frac{dy}{dx}=-\frac{1}{x^2-1}.$$

(*v*) Let $y=\operatorname{sech}^{-1} x$, or $x=\operatorname{sech} y$

$$\therefore \quad \frac{dy}{dx}=-\frac{1}{x\sqrt{1-x^2}}.$$

(*vi*) Let $y=\operatorname{cosec}h^{-1} x$, or $x=\operatorname{cosec}h\ y$

$$\therefore \quad \frac{dy}{dx}=-\frac{1}{x\sqrt{x^2+1}}.$$

Note. It should be noted that $\sinh x$, $\cosh x$, $\tanh x$, are continuous for all values of x and $\sinh^{-1} x$ is defined for all real values of x and is one-valued; $\cosh^{-1} x$ is defined only for $x\geqslant 1$ and is, in general, two valued; $\tanh^{-1} x$ is defined for $-1<x<1$ and is one valued.

EXAMPLES VI

1. Prove that each of the functions $\sec^{-1}(\cosh x)$, $\tan^{-1}(\sinh x)$, $\cos^{-1}(\operatorname{sech} x)$, has the derivative $\operatorname{sech} x$.

2. Prove that each of the functions $2\tanh^{-1}\left(\tan\frac{x}{2}\right)$ $\cosh^{-1}(\sec x)$ has the derivative $\sec x$.

Find the derivatives of the following functions :

3. $\dfrac{\cosh x+\cos x}{\sinh x+\sin x}$, $\tanh^{-1}\dfrac{3x+x^3}{1+3x^2}$, $\tanh^{-1}\dfrac{2x}{1+x^2}$.

4. $\log\cosh x$, $\log\tanh x$, $\log\operatorname{cosec}h\ x$, $a^{\sinh x}$.

5. $\sinh x+\frac{1}{3}\sinh^3 x$, $\sinh^{-1}\sqrt{x^2-1}$.

6. $\log\dfrac{1+\sqrt{1-x^2}}{x}$, $\log\dfrac{1+\sqrt{1+x^2}}{x}$.

7. $\tanh^{-1}\dfrac{x^2-1}{x^2+1}$, $\log\tanh x$.

8. $\log\left(x\sqrt{x^2+a^2}\right)$.

9. If $y=\dfrac{\sin^{-1} x}{\sqrt{1-x^2}}$, prove that $(1-x^2)\ y_1=xy+1$.

6. Logarithmic Differentiation.

The method of logarithmic differentiation can sometimes be employed. This method rests on the assumption that $\frac{dy}{dx}$ exists.

This method is applied to the differentiation of a function of the type $y=\frac{u_1u_2u_3\ldots u_n}{v_1v_2v_3\ldots v_m}$ or $y=u^v$ or $y=(\log u)^w$ where u's, v's, w's are all functions of x.

Let $y=u^v$

Then $\log y=v\log u$.

Therefore $\frac{1}{y}\frac{dy}{dx}=v\frac{1}{u}\cdot\frac{du}{dx}+\frac{dv}{dx}\log u$

or $\frac{dy}{dx}=y\left(\frac{v}{u}\frac{du}{dx}+\frac{dv}{dx}\log u\right)$.

Ex. 1. Differentiate $y=\frac{(x+1)^{\frac{3}{4}}(x^2+3)^{\frac{5}{3}}(x^3+7)^7}{\sqrt{x}(\sqrt{x}+2)^{\frac{4}{3}}}$.

$\log y=\frac{3}{4}\log(x+1)+\frac{5}{3}\log(x^2+3)+7\log(x^3+7)-\frac{1}{2}\log x-\frac{4}{3}\log(\sqrt{x}+2)$.

$\therefore\quad\frac{1}{y}\frac{dy}{dx}=\frac{3}{4}\frac{1}{x+1}+\frac{5}{3}\cdot\frac{2x}{x^2+3}+\frac{21x^3}{x^3+7}-\frac{1}{2x}$

$$-\frac{2}{3}\frac{1}{\sqrt{x}(\sqrt{x}+2)}.$$

This gives $\frac{dy}{dx}$.

(ii) Let $y=u^v$. Then

$\log y=v\log u$.

$\therefore\quad\frac{1}{y}\frac{dy}{dx}=\frac{v}{u}\frac{du}{dx}+\log u\frac{dv}{dx}$

$\therefore\quad\frac{dy}{dx}=vu^{v-1}\frac{du}{dx}+u^v\log u\frac{dv}{dx}$.

Ex. 2. Differentiate x^x.

Let $y=x^x$, then $\log y=x \log x$

$$\therefore \quad \frac{1}{y}\frac{dy}{x}=\log x+1=\log ex.$$

$$\therefore \quad \frac{dy}{dx}=x^x \log ex.$$

(*iii*) Let $y=(\log_v u)^w=\left(\frac{\log_e u}{\log_e v}\right)^w$

$$\log y=w \log\left\{\frac{\log u}{\log v}\right\}=w \log \log u-w \log \log v$$

$$\therefore \quad \frac{1}{y}\frac{dy}{dx}=\log \log u \frac{dw}{dx}+\frac{w}{u \log u}\frac{du}{dx}$$
$$-\log \log v \frac{dw}{dx}-\frac{w}{v \log v}\frac{dv}{dx}.$$

This gives dy/dx.

EXAMPLES VII

Differentiate the following with respect to x :

1. $\log_{\cos x} \sin x$.

2. x^{1+x}.

3. $\frac{(x+1)(x+2)(x+3)(x+4)}{(x-1)(x-2)(x-3)(x-4)}$ . .

4. $\sqrt{x}\sqrt{\sin x}\sqrt{\log x}$.

5. $(\tan x)^{\cot x}$.

6. $(\sin x)^{\log x}$.

7. $x^{\sin x}$.

8. $(\sin^{-1}x)^x$.

9. $(\sec x)^{\operatorname{cosec} x}+(\operatorname{cosec} x)^{\sec x}$

10. Find $\frac{dy}{dx}$ when $(\tan x)^y=(\tan y)^x$.

11. If $x^m y^n=(x+y)^{m+n}$ prove that $\frac{dy}{dx}=\frac{y}{x}$.

EXAMPLES OF DIFFERENT TYPES

Ex. 1. Differentiate $\tan^{-1}\frac{a+\beta x}{\beta-ax}$, a, β being constants.

Let $y=\tan^{-1}\frac{a+\beta x}{\beta-ax}$ and put $x=\tan\theta$. $\frac{a}{\beta}=\tan\lambda$.

Then $y=\tan^{-1}\dfrac{\tan\lambda+\tan\theta}{1-\tan\lambda\tan\theta}=\tan^{-1}\tan(\lambda+\theta)=\lambda+\theta$

$$\therefore \quad \frac{dy}{dx}=\frac{d\theta}{dx}=\frac{d}{dx}(\tan^{-1}x)=\frac{1}{1+x^2}.$$

Ex. 2. Find $\dfrac{dy}{dx}$ when $y=\sin^{-1}(\sqrt{x-ax}-\sqrt{a-ax})$.

Put $\sqrt{x}=\sin\theta$. $\sqrt{a}=\sin\alpha$. Then

$$y=\sin^{-1}[\sin\theta\cos\alpha-\cos\theta\sin\alpha]$$
$$=\sin^{-1}[\sin(\theta-\alpha)]=\theta-\alpha$$

$$\therefore \frac{dy}{dx}=\frac{d\theta}{dx}=\frac{1}{2\sqrt{x}\cos\theta}=\frac{1}{2\sqrt{x-x^2}}.$$

Ex. 3. Differentiate x^9 with regard to x^4.

Now $\dfrac{dy}{dx}=9x^8$, $\dfrac{du}{dx}=4x^3$.

$$\therefore \quad \frac{dy}{du}=\frac{dy}{dx}\cdot\frac{dx}{du}=9x^8\div 4x^3=\tfrac{9}{4}x^5.$$

Ex. 4. Differentiate $\tan^{-1}\dfrac{2x}{1-x^2}$ with regard to $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$

Let $y=\tan^{-1}\dfrac{2x}{1-x^2}$ and $z=\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x}$.

We have to find $\dfrac{dy}{dz}$.

Now $y=\tan^{-1}\dfrac{2x}{1-x^2}=\tan^{-1}(\tan 2\theta)$, (where $x=\tan\theta$)

$$=2\theta=2\tan^{-1}x;$$

and $$z=\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}=\tan^{-1}\frac{\sec\phi-1}{\tan\phi}\ (\text{where } x=\tan\phi)$$

$$=\tan^{-1}\frac{1-\cos\phi}{\sin\phi}=\tan^{-1}\left(\tan\frac{\phi}{2}\right)=\tfrac{1}{2}\tan^{-1}x.$$

$\therefore\ y=4z$, whence $\dfrac{dy}{dz}=4$.

Ex. 5. Find $\dfrac{dy}{dx}$ when $x=a(1-\cos\theta)$, $y=a(\theta+\sin\theta)$.

Here $\dfrac{dx}{dx}=\dfrac{dy}{d\theta}\cdot\dfrac{d\theta}{dx}=\dfrac{dy}{d\theta}\Big/\dfrac{dx}{d\theta}$.

But $\dfrac{dy}{d\theta}=a\sin\theta$ and $\dfrac{dy}{d\theta}=a(1+\cos\theta)$,

$\therefore\quad \dfrac{dy}{dx}=\dfrac{a(1+\cos\theta)}{a\sin\theta}=\dfrac{2\cos^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}$

$=\cot\frac{1}{2}\theta$

Ex. 6. $f(x)=x^2\sin\dfrac{1}{x}$ when $x\neq0$ and $f(0)=0$, find $f'(x)$.

Can we deduce $f'(0)$?

Let $y=f(x)$.

(i) $\dfrac{dy}{dx}=f'(x)=2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}$ for $x\neq0$.

Now $\underset{x\to0}{\text{Lt}}\ f'(x)$ does not exist.

Again from the 1st principles

$$f'(0)=\text{Lt}\frac{\delta y}{\delta x}=\underset{\delta x\to0}{\text{Lt}}\,\delta x\sin\frac{1}{\delta x}=0$$

$\therefore\quad f'(0)=0.$

Hence we cannot deduce $f'(0)$ from $f'(x)$.

The student should note carefully that

$f'(a)=\underset{x\to0}{\text{Lt}}\ f'(x)$ only when $f'(x)$ is continuous at a.

Ex. 7. Find the derivative of $\sqrt{\sin x}$ from first principles.

Here $\dfrac{dy}{dx}=\underset{h\to0}{\text{Lt}}\dfrac{\sqrt{\sin(x+h)}-\sqrt{\sin x}}{h}$

$=\underset{h\to0}{\text{Lt}}\dfrac{\sin(x+h)-\sin x}{\{h\sqrt{\sin(x+h)}+\sqrt{\sin x}\}}$

$$= \lim_{h\to 0} \frac{2\cos(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h\{\sqrt{\sin(x+h)}+\sqrt{\sin x}\}}$$

$$= \lim_{h\to 0} \frac{\cos(x+\frac{1}{2}h)}{\sqrt{\sin(x+h)}+\sqrt{\sin x}} \cdot \frac{\sin\frac{1}{2}h}{\frac{1}{2}h}$$

$$= \frac{\cos x}{2\sqrt{\sin x}}. \qquad \left[\because \lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1.\right.$$

MISCELLANEOUS EXAMPLES

Differentiate the following:

1. $x+\frac{1}{x}$. **2.** $\frac{a+bx}{c+dx}$ **3.** $\sin\sqrt{x}$. **4.** $\sin(a+bx^n)$.

5. (*i*) $\sin x^\circ$. (*ii*) $\sec x^\circ$. **6.** $\sin^p x^q$. **7.** $\sin^{-1}x^2$.

8. (*i*) $\sin^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)$ (*ii*) $\sin^{-1}\frac{x^2}{\sqrt{x^4+a^4}}$.

9. (*i*) $\tan^{-1}\frac{1}{\sqrt{x^2-1}}$. (*ii*) $\tan^{-1}\frac{x\sin a}{1-\cos a}$.

(*iii*) $\tan^{-1}(\sin e^x)$.

10. $\tan^{-1}(\tanh x)$.

11. $\sin^{-1}\frac{a+b\cos x}{b+a\cos x}$.

12. (*i*) x^{11} with regard to x^8. (*ii*) $\sin x$ with regard to $\cos x$.

13. $x+\sin x$ with regard to $\log x$.

14. $\tanh^{-1}\frac{x^2-1}{x^2+1}$ with regard to $\log x$.

15. $\tan^{-1}\frac{3x-x^3}{x-3x^2}$ with regard to $\tan^{-1}\frac{4x-4x^3}{1-6x^2+x^4}$.

16. Differentiate (*i*) $\tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$ with respect to $\tan^{-1}x$.

(*ii*) $\tan^{-1}\frac{2x}{1-x^2}$ with respect to $\tan^{-1}\frac{2x}{1+x^2}$.

17. Find dy/dx in the following cases :

(*i*) $x=a\cos\theta$, $y=b\sin\theta$ (*ii*) $x=a\cos^3\phi$, $y=b\sin^3\phi$.

(*iii*) $x=a(\cos t+\log\tan\frac{t}{2})$, $y=a\sin t$.

18. $\cos^{-1}(4x^3-3x)$.

19. $\tan^{-1}\left[\frac{1-\cos x}{1+\cos x}\right]^{\frac{1}{2}}$

20. $\tan^{-1}\frac{\sqrt{x}-x}{1+x^{3/2}}$.

21. $\cot^{-1}\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}$.

22. $\tan^{-1}\frac{\sqrt{1+x^2}-\sqrt{1-x^2}}{\sqrt{1+x^2}+\sqrt{1-x^2}}$.

23. $\log\sqrt{\left\{\frac{\sqrt{1+x}+\sqrt{1-x}}{\sqrt{1+x}-\sqrt{1-x}}\right\}}$.

24. $\tan^{-1}\frac{x}{1+\sqrt{1-x^2}}+\sin\left\{2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right\}$.

25. Investigate a method of finding the derivative with respect to x of $[f(x)]^{\phi(x)}$ and apply it to differentiate $x^{\sqrt{1+x^2}}(\sin x)^{\sec x}$.

26. $(\sin x)^{\cos x}+(\cos x)^{\sin x}$

27. If $y=\cfrac{\sqrt{x}}{1+\sqrt{x}}$

$$\cfrac{}{1+\sqrt{x}}$$

$$\cfrac{}{1+\text{etc, ad infinitum,}}$$

show that $\dfrac{dy}{dx}=\dfrac{1}{2\sqrt{x}\sqrt{1+4\sqrt{x}}}$.

28. $y=\cfrac{x}{a+}\ \cfrac{x}{b+}\ \cfrac{x}{a+}\ \cfrac{x}{b+}\ \ldots\ldots\text{ad inf.}$

29. $y=x+\cfrac{1}{x+}\ \cfrac{1}{x+}\ \cfrac{1}{x+}\ \ldots$

30. $y=\sqrt{\tan x+\sqrt{\tan x+\sqrt{\tan x+}}}\ldots\ldots\text{ad inf.}$

31. $y=(\cos x)^{(\cos\ x)^{(\cos x)}}$ ....ad inf.

32. $y=x^{x^{x}}$ .....a l infinitum.

33. If $y=\sqrt{\sin x+\sqrt{\sin x+\sqrt{\sin x+\sqrt{\text{etc. to }\infty}}}}$,

show that $\frac{dy}{dx}=\frac{\cos x}{2y-1}$.

34. Find $\frac{dy}{dx}$ when $y^x+x^y=a$.

Find $\frac{dy}{dx}$ when y equals x.

Differentiate :

35. $\frac{e^{x^2}\tan^{-1}x}{\sqrt{1+x^2}}$.

36. $\log\left\{e^x\left(\frac{x-2}{x+2}\right)^{\frac{3}{4}}\right\}$.

37. $\log\left(\frac{1+\sqrt{x}}{1-\sqrt{x}}\right)$.

38. $\log\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}$.

39. $e^{at}\cos\ (b\tan^{-1}t)$.

40. If $u=\sin^{-1}\frac{x\sqrt{2}}{1+x^2}$ and $v=\log\frac{\sqrt{1+x^4}+x\sqrt{2}}{1-x^2}$, prove

that $2\frac{dx}{du}\cdot\frac{dx}{dv}=1+x^4$.

41. Find the derivative to the right and to the left at $x=0$ of the function $f(x)=x\tan^{-1}\frac{1}{x}$ for $x\neq 0$ and $f(0)=0$.

42. A function $f(x)$ is defined by saying that it equals 0, x or $-x$ according as x is 0, >0 or <0. Prove that $f'(0)$ does not exist.

43. Explain what is meant by the continuity of a function at a point. Is the function

$f(x)=(x-a)\sin\frac{1}{x-a}$ for $x\neq a$;

$f(a)=0$

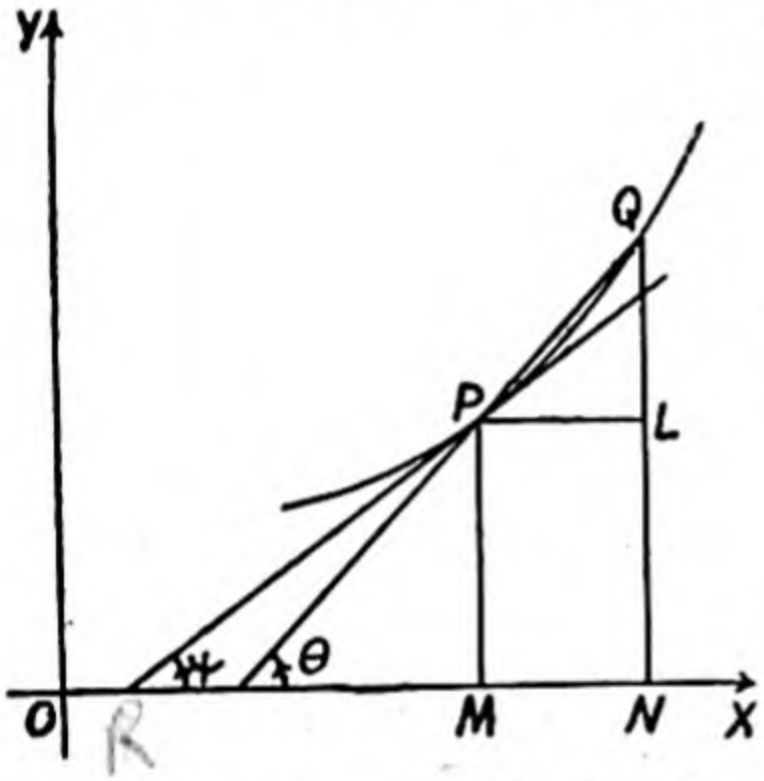
continuous and differentiable at $x=a$? Give your answer with reasons.

44. Find from first principles the derivatives of :

(*i*) $\sin x^2$ (*ii*) $e^{\sqrt{x}}$ (*iii*) $\dfrac{x^2+2}{x^3+3}$ (*iv*) $\log \sin x$.

7. Geometrical significance of the derivative.

Let $P(x, y)$ and $Q(x+\delta x,\ y+\delta y)$ be points on the curve $y=f(x)$. Draw PM,QN perpendiculars to the x-axis and PL $\perp$ QN.



Then $PL=\delta x$, $LQ=\delta y$.

If the secant PQ makes an angle θ with the x-axis, then

$$\frac{\delta y}{\delta x}=\frac{LQ}{PL}=\tan\theta.$$

Thus the incremental ratio is the tangent of the angle which the secant makes with the x-axis.

Suppose now that y is continuous in a small interval about x. When $\delta x\to 0$, Q approaches the fixed point P from either side and if the secant PQ tends to a limit position PR, we say that PR is the tangent to the curve at P.

Let the tangent at P make anangle ψ with the x-axis, then if $f'(x)$ is finite,

$$f'(x)=\text{Lt}\ \frac{\delta y}{\delta x}=\text{Lt}\ \tan\theta=\tan\psi.$$

Hence $f'(x)$ *represent the slope of the tangent to the curve at the point P.*

8. Properties of the derivative.

We consider a function $\phi(x)$ which has a derivative continuous in a given interval (a, b).

I. Meaning of the Sign of the Derivative.

(*a*) If $\phi'(x)>0$ for all values of x in the interval, $\phi(x)$ is an increasing function of x throughout the interval.

(*b*) If $\phi'(x)<0$ for all values of x in the interval, $\phi(x)$ is a decreasing function of x throughout the interval.

(*c*) If $\phi(x)$ is increasing in the interval, $\phi'(x) \geq 0$.

(*d*) If $\phi(x)$ is decreasing in the interval, $\phi'(x) \leq 0$.

We note that the existence of $\phi'(x)$ in the interval implies that $\phi(x)$ is continuous in the interval.

We have

$$\frac{\delta y}{\delta x} = \phi'(x) + a,$$

where a tends to 0 with δx.

(*a*) If $\phi'(x)>0$, then since $a \to 0$, we can take δx so small that $|a| < \phi'(x)$ so that $\phi'(x)+a>0$.

Thus we can find ϵ such that δy will have the same sign as δx for all $|\delta x| < \epsilon$.

This is true for all points in the range.

Hence $\phi(x)$ is increasing in the interval.

(*b*) If $\phi'(x)<0$, we have similarly $\phi'(x)+a<0$.

Thus δy will have the opposite sign to δx for all $|\delta x| < \epsilon$.

This is true for all points in the range.

Hence $\phi(x)$ is decreasing in the range.

(*c*) If $\phi(x)$ is increasing in the interval, then δy has the same sign as δx,

$$\therefore \quad \frac{\delta y}{\delta x} > 0$$

Hence Lt $\frac{\delta y}{\delta x} \geq 0$, *i. e.*, $\phi'(x) \geq 0$.

(*d*) If $\phi(x)$ is decreasing in the interval, then δy has the opposite sign to δx.

$$\therefore \quad \frac{\delta y}{\delta x} < 0.$$

Hence Lt $\frac{\delta y}{\delta x} \leq 0$, *i. e.*, $\phi'(x) \leq 0$.

Cor. 1. If $\phi'(x)>0$ throughout the interval (a, b) and $\phi'(a)\geq 0$, then $\phi(x)>0$ throughout the interval.

Cor. 2. If $\phi'(x)<0$ throughout the interval (a, b) and $\phi(a)\leq 0$, then $\phi(x)<0$ throughout the interval.

Ex. Show that $\phi(x)=x^3$ is increasing in the interval $(-1, +1)$ and $\phi(x)=0$ at $x=0$.

It follows that if the derivative of $f(x)$ is not zero at a point $x=a$ the function always takes, in the immediate neighbourhood of that point, values greater than as well as less than the value $f(a)$ it takes at that point. For example it takes greater values to the right of $x=a$ if the function is increasing and to the left of $x=a$ if the function is decreasing ; similarly it takes less values to the left of $x=a$ if the function is increasing and to the right of $x=a$ if the function is decreasing. Consequently if $f'(a)$ exists and is not zero, then $f(a)$ is neither the greatest nor the least value of $f(x)$ in the immediate neighbourhood of $x=a$, that is in the interval $(a-\delta, a+\delta)$, however small δ may be.

II. Rolle's Theorem.

If the function $f(x)$ is continuous in (a, b) and possesses a derivative in the open interval (a, b) that is the derivatives at a and b may not exist and $f(a)=0$, and $f(b)=0$, then there must be at least one value of x between a and b for which $f'(x)=0$.

It is given that $f(a)=0$, $f(b)=0$. If the function is 0 throughout the interval, then it is constant in that interval and consequently its derivative is 0 throughout and the theorem is proved.

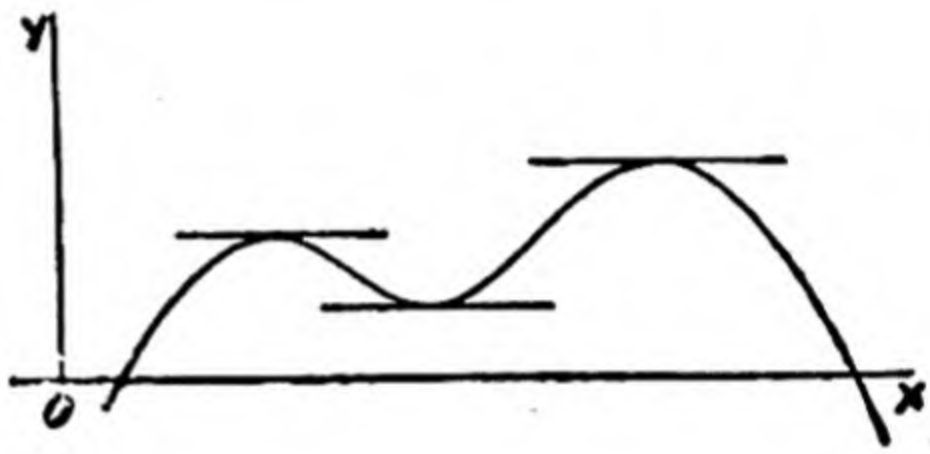
If the function is not zero throughout, then at least one of its bounds is different from 0. Let its upper bound (or its lower bound as the case may be) be different from zero. This bound is attained for at least one value of x, say $x=c$, so that $f(c)$ is its upper bound (or lower bound as the case may be). Then $f'(c)=0$. For if $f'(c)$ is not zero, then $f(x)$ is either increasing or decreasing at $x=c$ and in either case it takes values greater than as well as less than the value $f(c)$ in the immediate neighbourhood of $x=c$; and consequently $f(c)$ cannot be its upper bound (or lower bound as the case may be).

Cor. If $f(a)=f(b)$ and the function possesses a derivative in (a, b) then there is at least one value of x between a and b for which $f'(x)=0$.

Consider the function $\phi(x) \equiv f(x)-f(a)$.

$\phi(a)=0$ and $\phi(b)=0$ and consequently there is at least one value of x, say c, for which $\phi'(c)=0$. But $\phi'(x)=f'(x)$ and consequently $f'(c)=0$.

Geometrical Illustration. Draw the graph of the function $f(x)$ from $x=a$ to $x=b$. The graph is a continuous curve having a tangent everywhere in the intervals. The theorem says that there must be some point P on the curve the tangent at which is parallel to the x-axis.



III. Mean Value Theorem.

Let $f(x)$ be a function continuous in (a, b) and possessing a derivative in the open interval (a,b); there exists a point c between a and b such that

$$f(b)-f(a)=(b-a)\,f'(c), \text{ where } a<c<b.$$

Consider the function

$$\phi(x)=(b-a)\{f(x)-f(a)\}-(x-a)\{f(b)-f(a)\}.$$

From the given conditions it follows that $\phi(x)$ is continuous in (a, b), possesses a derivative and $\phi(a)=0$ and $\phi(b)=0$. Therefore by Rolle's Theorem there exists a value of x(say c) between a and b such that $\phi'(c)=0$.

But $\phi'(x)=(b-a)f'(x)-\{f(b)-f(a)\}$

Therefore $\phi'(c)=0$ gives us

$$(b-a)f'(c)-\{f(b)-f(a)\}=0.$$

or $f(b)-f(a)=(b-a)f'(c)$, $a<c<b$.

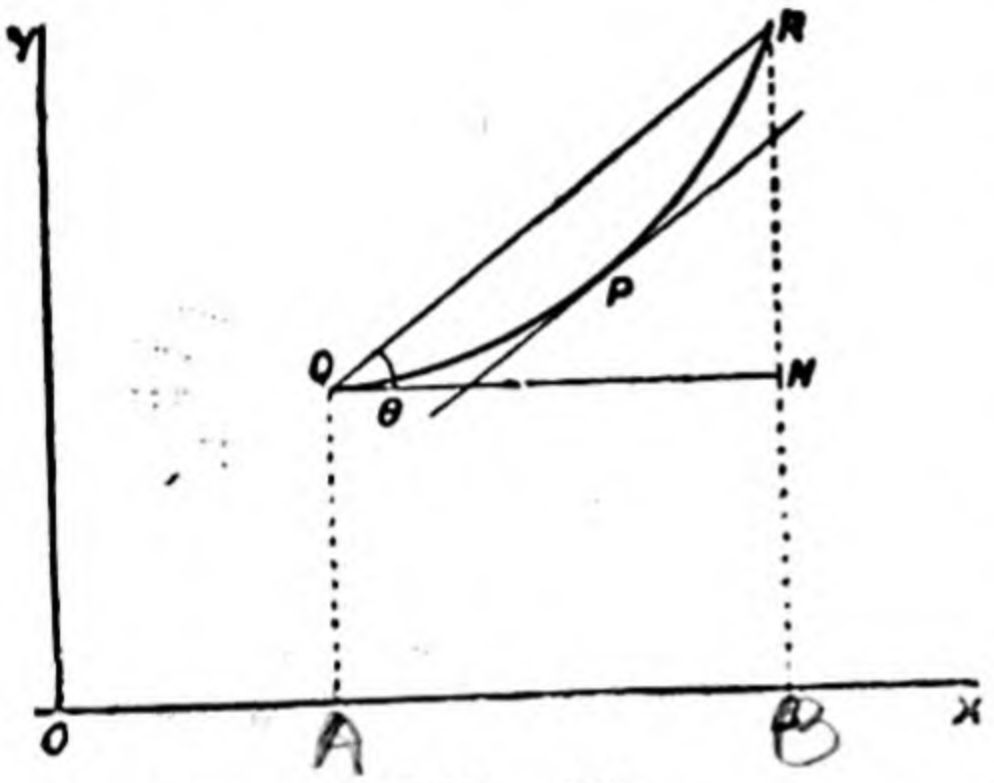
The relation is commonly written as follows

Let $b-a=h$ or $b=a+h$; then since c lies between a and b, that is, between a and $a+h$, $c=a+\theta h$ where θ is a positive proper fraction (or $0<\theta<1$).

Thus the above relation becomes

$$f(a+h)=f(a)+hf'(a+\theta h),\ 0<\theta<1.$$

Note. The **geometrical meaning** of the theorem is as follows :



Let QPR be the graph of $f(x)$ in the interval (a, b).

Let the chord RQ make an angle θ with the x-axis.

OA$=a$, OB$=b$,

AQ$=f(a)$, BR$=f(b)$

$$\tan\theta=\tan \text{ NQR}=\frac{f(b)-f(a)}{b-a}.$$

The theorem asserts that there is some point P between Q and R, where the tangent to the curve $y=\phi(x)$ is parallel to the chord QR, *i.e.*, if a curve has a continuous tangent at every point of the arc concerned, there is in the arc cut off by any chord some point at which the tangent is parallel to the chord.

IV. If the derivative of $f(x)$ is zero at all points of an interval, the function is a constant throughout that interval.

For, if x_1 and x_2 be any two values of x in the interval, then

$f(x_2)-f(x_1)=(x_2-x_1)f'(c)$, where c lies between x_1 and x_2

$=0$;

because the derivative is given to be zero.

Therefore $f(x_2)=f(x_1)$.

Hence $f(x)$ is a constant.

Cor. If two functions $f(x)$ and $\phi(x)$ have equal derivatives in (a, b), then the functions differ by a constant throughout the interval.

For the derivative of the function $f(x)-\phi(x)$ being $f'(x)-\phi'(x)$ is zero throughout the interval and therefore the function $f(x)-\phi(x)$ is a constant in that interval.

V. Cauchy's Theorem.

Let $\phi(x)$ and $f(x)$ be two functions continuous in (a, b) and possessing finite derivatives $\phi'(x)$ and $f'(x)$ and let $\phi'(x)$ be not zero in the interval. Then

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)}=\frac{f'(c)}{\phi'(a)}, \text{ where } a<c<b.$$

It may be observed that $\phi(b)$ is not equal to $\phi(a)$, for if $\phi(b)=\phi(a)$ then $\phi'(x)=0$ for some value of x lying between a and b, which is contrary to the hypothesis.

Consider the function

$$F(x) \equiv \{\phi(b)-\phi(a)\}\{f(x)-f(a)\}-\{f(b)-f(a)\}\{\phi(x)-\phi(a)\}$$

$F(x)$ is obviously continuous in the interval and possesses a finite derivative ; and also $F(a)=0$, $F(b)=0$.

Therefore by Rolle's Theorem $F'(c)=0$ for $a<c<b$.

But $F'(x)=\{\phi(b)-\phi(a)\}f'(x)-\{f(b)-f(a)\}\phi'(x)$

and therefore $F'(c)=0$ becomes

$$\{\phi(b)-\phi(a)\}f'(c)=\{f(b)-f(a)\}\phi'(c).$$

Dividing by $\phi'(c)$ and $\{\phi(b)-\phi(a)\}$, we get

$$\frac{f(b)-f(a)}{\phi(b)-\phi(a)}=\frac{f'(c)}{\phi'(c)}.$$

Ex. 1. Show that $\frac{\sin x}{x}$ continually decreases in the interval $\left(0, \frac{\pi}{2}\right)$.

Let $\phi(x)=\frac{\sin x}{x}$.

Then $\phi'(x)=\dfrac{x\cos x-\sin x}{x^2}=\dfrac{\cos x(x-\tan x)}{x^2}$.

Since $\phi(x)$ and $\phi'(x)$ are continuous and $\phi'(x)<0$ $(x<\tan x)$ in the interval $\left[0, \dfrac{\pi}{2}\right]$.

$\therefore$ $\phi(x)$ is decreasing.

Ex. 2. If $x>0$, prove that $x-\log(1+x)>\dfrac{\frac{1}{2}x^2}{1+x}$.

Let $\phi(x)=x-\log(1+x)-\frac{1}{2}\dfrac{x^2}{1+x}$

$\therefore$ $\phi'(x)=1-\dfrac{1}{1+x}-\frac{1}{2}\dfrac{(1+x)2x-x^2}{(1+x)^2}=\dfrac{x^2}{2(1+x)^2}>0.$

$\therefore$ $\phi'(x)$ is positive in the interval $(0, \infty)$.

$\therefore$ $\phi(x)$ is increasing in the interval.

But $\phi(0)=0$. Hence $\phi(x)>0$.

EXAMPLES VIII

1. In the theorem $f(b)-f(a)=(b-a)f'(c)$, if $f(x)=x^2-3x-1$, $a=1$, $b=3$, find c.

2. Show that $\dfrac{\tan x}{x}$ continually increases in the interval $[0, \dfrac{\pi}{2}]$.

3. If $y=2x^3-9x^2+12x-6$, find the range of values of x for which y is increasing and that for which it is decreasing.

4. Verify the Mean Value Theorem for $\phi(x)=x^3$ in the interval (3,4).

5. Show that $e^x(x-2)+x+9$ is positive for all positive values of x.

6. Prove that $(x+2)\log(1+x)>2x$ for $x>0$.

7. Show that $x^3-6x^2+15x+3$ is always positive if x be positive.

8. Show that sin x lies between $x-\dfrac{x^3}{6}$ and $x-\dfrac{x^3}{6}+\dfrac{x^5}{120}$

9. Show that $\log(1+x)$ lies between $x-\frac{x^2}{2}$ and $x-\frac{x^2}{2(1+x)}$ for $x>0$.

10. Show that

$$x-1>\log x>\frac{x-1}{x}$$

and $x^2-1>2x\log x>4(x-1)-2\log x$ for $x>1$.

11. If $\phi(x)=0$ has two equal roots, show that $\phi'(x)=0$ has one root equal to either.

12. Show that if a root a is repeated r times in the equation $f(x)=0$, then it is repeated $(r-1)$ times in $f'(x)=0$.

13. Determine α, β, γ, δ, so that the curve

$$y=\alpha x^3+\beta x^2+\gamma x+\delta$$

may pass through the point $(-1,4)$, touch the x-axis at the point $(3,0)$ and have its tangent parallel to the x-axis at the point for which $x=1$.

14. Find a point on the curve $y=x^3$ at which the tangent is parallel to the chord through the point (1, 1), (2, 8).

15. Differentiate a determinant of the third order whose constituents are differentiable functions of x.

[*Differentiate the constituents of each row (or column) and leave those of the other rows (or columns) untouched and then add all the determinants so obtained.*]

16. Determine the constants A and B such that $Af(x)+B\,\phi(x)$ has the same value for $x=a$ and $x=a+h$. Hence show that under certain conditions

$$\frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)}=\frac{f'(a+\theta h)}{\phi'(a+\theta h}, \quad 0<\theta<1.$$

17. Show that if $f''(x)$ exists in the open interval (a, b), then

$f(x+h)+f(x-h)-2f(x)=h^2 f''(x+\theta h)$, when θ is some number between -1 and 1.

18. Discuss the applicability of Rolle's Theorem in (0, 2) to the function $f(x)=2+(x-1)^{\frac{2}{3}}$. Illustrate your answer by a rough sketch.

CHAPTER III

SUCCESSIVE DIFFERENTIATION

1. Successive Derivatives. Let $y=f(x)$ be a function of x having a derivative in an interval. If that new function, $y'=f'(x)$, possesses a derivative at a point x, that new derivative is called the *second* derivative of y and is represented by $\frac{d^2y}{dx^2}$, y'', y_2, $f''(x)$, $f_2(x)$, or D^2y. In the same way, the *third* derivative of y is the derivative of y'' and is denoted by $\frac{d^3y}{dx^3}$, y''', y_3, $f'''(x)$ $f_3(x)$or D^3y. Continuing, the nth derivative (or the *derivative of order* n)of y is the derivative of that of order $(n-1)$ and is denoted by $\frac{d^ny}{dx^n}$ y_n, $f^n(x)$, $f^n(x)$ or D^ny.

It may be observed that the existence of the derivative of order n at a point x requires that the derivative of order $n-1$ must exist and be finite in the vicinity of the point x and must be continuous at that point, and consequently all the derivatives of lower order must be continuous at and in the vicinity of that point.

Ex. 1. If $y=x^4$, find y_1, y_2, y_3, y_4 ····

Here $y_1=4x^3$, $y_2=4.3.x^2$, $y_3=4.3.2x$

$y_4=4.3.2.1$, $y_5=0=y_6=y_7=$ ····

Ex. 2. If $y=\log \sin x$, find y_3.

Here $y_1=\cot x$, $y_2=-\operatorname{cosec}^2x$,

and $y_3=+2\operatorname{cosec}^2x \cot x$.

2. Successive Differentials. If the differential dy of the function $y=f(x)$ is differentiable then the differential of dy is called the second differential of $y=f(x)$ and is denoted by d^2y.

This new differential depends on the relation which we wish to establish between the variable x and its difference δx or dx. If δx or dx is the same for all values of x and is also the same for each successive differentiation, we may regard

as a constant and then we have

$$dy=f'(x)\,dx$$
$$d^2y=d[f'(x)dx]=\{f''(x)\,dx\}dx=f''(x)dx^2$$
$$d^3y=d[f''(x)dx^2]=\{f'''(x)\,dx^2\}dx=f'''(x)dx^3.$$
$$\ldots\ldots\ \ldots\ldots\ \ldots\ldots\ldots\ldots$$
$$d^ny=d[f^{n-1}(x)dx^{n-1}]=\{f^n(x)dx^{n-1}\}dx=f^n(x)dx^n.$$

From these equations it follows that

$$f'(x)=\frac{dy}{dx},\ f''(x)=\frac{d^2y}{dx^2}\ldots\ldots\ f^n(x)=\frac{d^ny}{dx^n}.$$

Therefore the nth derivative of a function is the quotient of the nth differential of the function by the nth power of the differential of the variable.

3. Standard Results. The determination of a derivative of any order for a function does not present any difficulty, except that the calculations may be lengthy. But the problem of finding the nth derivative of a function, n being arbitrary, is more difficult. However, for elementary functions, the solution is quite easy.

I. Find the nth derivative of $(ax+b)^m$.

Here $y=(ax+b)^m$

$$y_1=ma(ax+b)^{m-1}$$
$$y_2=m(m-1)\,a^2(ax+b)^{m-2}$$
$$\ldots\ldots\ldots\ldots$$
$$y_n=m(m-1)\ldots\ldots(m-n+1)a^n(ax+b)^{m-n}.$$

Suppose m is a positive integer; this derivative reduces to the constant $\lfloor m$ if $n=m$, and that is zero if $n>m$. Therefore the nth derivative of a polynomial of degree less than n is identically zero.

II. Find the nth derivative of $\log(ax+b)$.

Here $y=\log(ax+b)$

$$y_1=\frac{a}{ax+b}=a(ax+b)^{-1}$$
$$y_2=(-1)\,a^2(ax+b)^{-2}$$

$y_2=(-1)(-2)a^3(ax+b)^{-3}$

.....

$y_n=(-1)(-2).....(-n+1)a^n(ax+b)^{-n}$

i. e., $\mathbf{y_n=(-1)^{n-1}\frac{\lfloor n-1}{(ax+b)^n}a^n}$

Cor. If $y=\frac{1}{ax+b}$, $y_n=\frac{(-1)^n\lfloor n}{(ax+b)^{n+1}}a^n$.

III. Find the nth derivative of $\mathbf{a^x}$.

Here $y=a^x$

$y_1=a^x \log a$

$y_2=a^x (\log a)^2$

.....

$\mathbf{y_n=a^x(\log a)^n}$.

Cor. If $y=e^x$, $y_n=e^x$.

IV. Find the nth derivative of $\mathbf{\sin(ax+b)}$.

Here $y=\sin(ax+b)$

$$y_1=a\cos(ax+b)=a\sin\left(ax+b+\frac{\pi}{2}\right)$$

$$y_2=a^2\cos\left(ax+b+\frac{\pi}{2}\right)$$

$$=a^2\sin\left(ax+b+2\frac{\pi}{2}\right)$$

Therefore $\mathbf{y_n=a^n\sin\left(ax+b+n\frac{\pi}{2}\right)}$.

Similarly if $\mathbf{y=\cos(ax+b)}$,

$$\mathbf{y_n=a^n\cos\left(ax+b+n\frac{\pi}{2}\right)}.$$

V. Find the nth derivative of $\mathbf{e^{ax}\sin(bx+c)}$.

Here $y=e^{ax}\sin(bx+c)$,

$y_1=ae^{ax}\sin(bx+c)+be^{ax}\cos(bx+c)$.

$$=e^{ax}[a \sin (bx+c)+b \cos (bx+c)]$$

Let $a=r \cos \theta$, $b=r \sin \theta$

so that $r^2=a^2+b^2$ and $\tan \theta=b/a$.

Therefore $y_1=re^{ax} \sin (bx+c+\theta)$.

Similarly $y_2=r^2 e^{ax} \sin (bx+c+2\theta)$,

and generally $\mathbf{y_n=r^n.e^{ax} \sin (bx+c+n\theta)}$.

It may be shown in the same way that if

$$\mathbf{y=e^{ax} \cos(bx+c)},\ \mathbf{y_n=r^n\ e^{ax} \cos(bx+c+n\theta)}.$$

It may be remarked that the reader who is acquainted with the Integral Calculus will observe that in the above examples if we take $n=-1$, we get the integral of the function concerned.

4. Sometimes preliminary simplifications render the problem of finding the nth derivative of a function quite easy.

Ex. 1. If $y=\sin^3 x \cos^2 x$, find y_n.

$$\text{Here } y=\frac{1}{4}\left(\frac{1-\cos 4x}{2}\right) \sin x$$

$$=\frac{1}{8}(\sin x-\sin x \cos 4x)$$

$$=\frac{1}{16}(2 \sin x-\sin 5x+\sin 3x)$$

$$\therefore\ y_n=\frac{1}{16}\left\{2 \sin\left(x+n\frac{\pi}{2}\right)-5^n \sin\left(5x+\frac{n\pi}{2}\right)\right.$$

$$\left.+3^n \sin\left(3x+n\frac{\pi}{2}\right)\right\}.$$

Ex. 2. If $y=e^x \sin^3 x$, find y_n.

$$\text{Here } y=e^x \sin^3 x=\frac{e^x(3 \sin x-\sin 3x)}{4}$$

$$=\frac{3}{4} e^x \sin x-\frac{1}{4} e^x \sin 3x$$

$$\therefore\ y_n=\frac{3}{4}\ 2^{\frac{n}{2}} e^x \sin\left(x+n\frac{\pi}{4}\right)$$

$$-\frac{1}{4}10^{\frac{n}{2}}e^{x}\sin(3x+n\tan^{-1}3).$$

EXAMPLES IX

Find the nth derivative in the following cases :—

1. $\frac{1}{1+x}$. **2** $\frac{1}{1-x}$. **3.** $\frac{1}{a-bx}$.

4. $\frac{1}{(x+a)^2}$. **5.** $\sqrt{x+a}$. **6.** $\sin x \sin 2x$.

7. $\cos x \cos 2x \cos 3x$. **8.** $\cos^4 x$ **9.** $e^x \cos^2 x$.

10. $e^x \sin x \sin 2x$. **11.** $e^{2x}\sin^2 x \cos^3 x$.

12. Find y_1, y_2, y_3 and y_4 when $y=x^3 \log x$.

13. Show that the equation

$$\frac{d^2s}{dt^2}+k\frac{ds}{dt}+n^2s=0$$

is satisfied by $s=Ae^{-\frac{k}{2}t}\cos(at+b)$, for all values of A and k provided that $a^2=n^2-\frac{1}{4}k^2$.

14. If $y=(a+bt)e^{-nt}$, show that

$$\frac{d^2y}{dt^2}+2n\frac{dy}{dt}+n^2y=0.$$

15. If $y=(A+Bx)\cos kx+(C+Dx)\sin kx$, prove that

$$\frac{d^4y}{dx^4}+2k^2\frac{d^2y}{dx^2}+k^4y=0.$$

16. $y=e^{-\frac{k}{2}x}(A\cos nx+B\sin nx)$. prove that

$$\frac{d^2y}{dx^2}+k\frac{dy}{dx}+\left(n^2+\frac{1}{4}k^2\right)y=$$

17. If $y=Ae^{-kt}\cos(pt+q)$, show that $\frac{d^2y}{dt^2}+2k\frac{dy}{dt}+n^2y=0$

where $n^2=p^2+k^2$.

18. If $y=e^{ax}\sin ax$, show that

$$\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+(a^2+b^2)y=0.$$

19. If $x=f(t)$, $y=\phi(t)$, prove that

$$\frac{d^2y}{dx^2}=\frac{\frac{dx}{dt}\cdot\frac{d^2y}{dt^2}-\frac{dy}{dt}\cdot\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}.$$

20. Find $\frac{d^2y}{dx^2}$ when $x=a(\theta-\sin\theta)$, $y=a(1-\cos\theta)$.

21. If $x=2\cos t-\cos 2t$ and $y=2\sin t-\sin 2t$, find $\frac{d^2y}{dx^2}$.

22. If $x=\sin t$, $y=\sin pt$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+p^2y=0.$$

23 If $ax^2+2hxy+by^2+2gx+2fy+c=0$, show that

$$\frac{d^2y}{dx^2}=\frac{abc+2fgh-af^2-bg^2-ch^2}{(hx+by+f)^3}.$$

[*Hint.* $\frac{dy}{dx}=-\frac{ax+hy+g}{hx+by+f}$;

$$\frac{d^2y}{dx^2}=-\frac{(ab-h^2)(ax^2+by^2+2hxy+2gx+2fy+c)-\Delta}{(hx+by+f)^3}$$

$$=\frac{\Delta}{(hx+by+f)^3}.$$

Observe that $x(ax+hy+g)+y(hx+by+f)+gx+fy+c$
$=ax^2+by^2+2hxy+2gx+2fy+c$.]

24. If $y=\sin mx+\cos mx$, prove that

$$y_n=m^n\{1+(-1)^n\sin 2mx\}^{\frac{1}{2}}.$$

5. Use of Partial Fractions. Assuming the decomposition of a rational function into partial fractions to be possible, its nth derivative can be easily found by first splitting it into partial fractions.

Ex. 1. Find the nth derivative of

$$\frac{x}{x^2-3x+2}.$$

Here $y=\frac{x}{x^2-3x+2}=\frac{\quad}{(x-2)(x-1)}$.

Let $$\frac{x}{(x-2)(x-1)} \equiv \frac{A}{x-2} + \frac{B}{x-1}.$$

Therefore $x \equiv A(x-1)+B(x-2)$.

In this identity, putting $x=1$ and $x=2$ successively, we get $B=-1$ and $A=2$.

Therefore $$y = \frac{2}{x-2} - \frac{1}{x-1},$$

and $$y_n = (-1)^n \underline{|n} \left(\frac{2}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}} \right).$$

Ex. 2. Find the nth derivative of

$$y = \frac{2x+1}{(x+2)^2(x-3)^2}.$$

Let $$\frac{2x+1}{(x+2)(x-3)^2} \equiv \frac{A}{x+2} + \frac{B}{x-3} + \frac{C}{(x-3)^2}.$$

Therefore $2x+1 \equiv A(x-3)^2+B(x-3)(x+2)+C(x+2)$.

Putting $x=3$ and $x=-2$ in succession, we get

$$C = \frac{7}{5} \quad \text{and} \quad A = -\frac{3}{25}.$$

Also comparing co-efficients of x^2 on both sides, we get

$$0 = A+B;$$

therefore $$B = \frac{3}{25}.$$

The nth derivative is therefore

$$y_n = -\frac{3}{25}\frac{(-1)^n \underline{|n}}{(x+2)^{n+1}} + \frac{3}{25}\frac{(-1)^n \underline{|n}}{(x-3)^{n+1}} + \frac{7}{5}\frac{(-1)^n \underline{|n+1}}{(x-3)^{n+2}}.$$

6. Use of DeMoivre's Theorem. When the denominator of a given algebraic fraction does not break up into real linear factors, the method of partial fractions can still be used by resolving the denominator into linear factors real or imaginary.

Ex. If $\mathbf{y = \frac{1}{x^3+a^3}}$, prove that

$$\mathbf{y_n = \frac{(-1)^n \,|\, n \;\sin^{n+1}\theta \sin(n+1)\theta}{a^{n+2}}}.$$

where $\mathbf{x = a\cot\theta}$.

Here $$y = \frac{1}{(x+ia)(x-ia)} = \frac{1}{2ia}\left(\frac{1}{x-ia} - \frac{1}{x+ia}\right)$$

$$\therefore \quad y_n = \frac{(-1)^n \,|\, n}{2ia}\left\{\frac{1}{(x-ia)^{n+1}} - \frac{1}{(x+ia)^{n+1}}\right\}.$$

Now let $x = r\cos\theta$ and $a = r\sin\theta$; then

$$y_n = \frac{(-1)^n \,|\, n}{2iar^{n+1}}\Big((\cos\theta - i\sin\theta)^{-(n+1)} - (\cos\theta + i\sin\theta)^{-(n+1)}\Big).$$

But by DeMoivre's Theorem

$$(\cos\theta - i\sin\theta)^{-n-1} = \cos(n+1)\theta + i\sin(n+1)\theta$$
and $$(\cos\theta + i\sin\theta)^{-n-1} = \cos(n+1)\theta - i\sin(n+1)\theta.$$

Therefore $$y_n = \frac{(-1)^n \,|\, n}{ar^{n+1}}\sin(n+1)\theta$$

$$= \frac{(-1)^n \,|\, n}{a^{n+2}}\sin(n+1)\theta\sin^{n+1}\theta.$$

This result is often wanted and the student should be able to obtain it immediately.

Note. If the denominator contains some imaginary linear factors, the nth derivative of the fraction may be reduced to *real form* by the aid of a form of DeMoivre's Theorem, namely, that

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta,$$
where n is a positive or negative integer.

EXAMPLES X

Find the nth derivative of the following:

1. $\dfrac{2x-3}{(x-2)(x+1)}$.
2. $\dfrac{x^3}{(x-3)(x+1)}$.
3. $\dfrac{1}{2-x-x^2}$.
4. $\dfrac{1}{(9x^2-1)}$.
5. $\dfrac{x^2+4x+1}{x^3+2x^2-x-2}$.
6. $\dfrac{1}{a^2x^2+1}$.

7. $\dfrac{1}{(x^2+a^2)(x^2+b^2)}$. 8. $\dfrac{1}{x^4-a^4}$. 9. $\dfrac{x^2}{1-x^2}$.

10. $\dfrac{1}{x^2+x+1}$. 11. $\dfrac{1}{x^3-x+1}$. 12. $\dfrac{x}{x^4+x^2+1}$.

13. $\dfrac{1}{x^3+1}$. 14. $\dfrac{1}{(x+1)(x^2+1)}$.

15. $\dfrac{x^2}{(x-1)^3(x+1)}$. 16. $\dfrac{x}{a^2+x^2}$. 17. $\dfrac{1}{(a^2+x^2)^2}$.

18. $\tan^{-1}\dfrac{x}{a}$. 19. $\tan^{-1}\dfrac{2x}{1-x^2}$. 20. $\tan^{-1}\dfrac{x\sin a}{1-x\cos a}$.

21. If $y=x\log\dfrac{x-1}{x+1}$ prove that

$$y_n=(-1)^n\,\underline{|n-2}\left(\frac{x-n}{(x-1)^n}-\frac{x+n}{(x+1)^n}\right).$$

7. Leibnitz's Theorem. The *n*th derivative of the product of two functions of x. *If u and v be functions of x having derivatives of any desired order, then,*

$$\frac{d^n}{dx^n}(uv)={}^nC_0\,u\,v_n+{}^nC_1\,u_1\,v_{n-1}+{}^nC_2\,u_2\,v_{n-2}$$
$$+\ldots\ldots+{}^nC_r\,u_r\,v_{n-r}+\ldots\ldots+{}^nC_n\,u_n\,v.$$

This theorem can be easily proved by the *method of Induction.*

Let us *assume* that the theorem holds for a *particular* number n, so that

$$D^n(uv)=u\,D^n v+{}^nC_1\,D\,u\,D^{n-1}v$$
$$+\ldots\ldots+{}^nC_{r-1}\,D^{r-1}u\,D^{n-r+1}v+{}^nC_r\,D^r u\,D^{n-r}v$$
$$+\ldots\ldots+v\,D^n u.$$

Then differentiating both sides of this equation, we get

$$D^{n+1}(uv)=u\,D^{n+1}v+(1+{}^nC_1)\,D\,u\,D^n v$$
$$+\ldots\ldots+({}^nC_{r-1}+{}^nC_r)\,D^r u\,D^{n-r+1}v+\ldots\ldots+v\,D^{n+1}u$$
$$=u\,D^{n+1}v+{}^{n+1}C_1\,D\,u\,D^n v$$
$$+\ldots\ldots+{}^{n+1}C_r\,D^r n\,D^{n+1-r}v+\ldots\ldots v\,D^{n+1}u.$$

Therefore, if Leibnitz's Theorem holds for any particular number n, it also holds for $n+1$.

But by *actual derivation,*

$$D\ (u\ v)=u\ v_1+v\ u_1,$$
and $D^2\ (u\ v)=u\ v_2+\ 2\ u_1v_1+v\ u_2.$

Hence the theorem holds for $n=2$ and therefore for $n=3$; and therefore for $n=4$ and so on; and hence the theorem is generally true

Ex. 1. Find the nth derivative of $x^3\ e^{ax}$.

Here let $u=x^3$ and $v=e^{ax}$.

$\therefore$ by Leibnitz's Theorem, we get

$$D^n(x^3\ e^{ax})=x^3\ a^n\ e^{ax}+{}^nC_1\ (3x^2)a^{n-1}\ e^{ax}$$
$$+{}^nC_2\ (6x)\ a^{n-2}\ e^{ax}+{}^nC_3\ (6)\ a^{n-3}\ e^{ax}$$
$$=e^{ax}\ a^{n-3}\ \{a^3\ x^3+3n\ a^2\ x^2+3n\ (n-1)xa$$
$$+n(n-1)(n-2)\}.$$

Ex. 2. Find the nth derivative of $x^2 \sin ax$.

Here let $u=x^2$, and $v=\sin ax$.

$$\text{Then } D^n\ (u\ v)=x^2a^n \sin\left(ax+\frac{n\pi}{2}\right)$$

$$+{}^nC_1\ 2x.a^{n-1} \sin\left(ax+\frac{n-1}{2}\pi\right)$$

$$+{}^nC_2\ 2.a^{n-2} \sin\left(ax+\frac{n-2}{2}\pi\right.$$

$$=a^n\ x^2 \sin\left(ax+\frac{n\pi}{2}\right)+2nxa^{n-1} \sin\left(ax+\overline{n-1}\ \frac{\pi}{2}\right)$$

$$+\frac{n(n-1)}{1.2}\ a^{n-2} \sin\left(ax+\overline{n-2}\ \frac{\pi}{2}\right).$$

Ex. 3 Differentiate n times the equation
$$(1-x^2)\ y_2-xy_1+m^3y=0.$$

$$D^n\ [(1-x^2)\ y_2]=(1-x^2)y_{n+2}+n(-2x)y_{n+1}+\frac{n(n-1)}{1.2}\ (-2)y_n$$

$$D^n\ (x\ y_1) \quad = \quad +x_{n+1} \quad +ny_n$$
$$D^n\ (m^2\ y) \quad = \quad m^2y_n$$

therefore the result is

$$(1-x^2)\ y_{n+2}-(2n+1)\ xy_{n+1}+(m^2-n^2)\ y_n=0.$$

8. Sometimes the general value of the nth derivative of

a function cannot be easily obtained; but we may be able to find its value for $x=0$.

Ex. Find the value of the nth derivative of

$$[\log (x+\sqrt{1+x^2}]^2$$

for $x=0$)

Let $y=\{\log (x+\sqrt{1+x^2}\}^2$

Then $$y_1=\frac{2\log (x+\sqrt{1+x^2})}{\sqrt{1+x^2}} \qquad (i)$$

and $(1+x^2)\ y_1^2+4y.=0$

Differentiating again, and dividing by $2y_1$, (provided that it is not zero) we get

$$(1+x^2)y_2+xy_1=2. \qquad (ii)$$

Differentiating this n times by Leibnitz's Theorem, we have

$$(1+x^2)y_{n+2}+2nx\ y_{n+1}+n(n-1)y_n+xy_{n+1}+ny_n=0,$$

or $(1+x^2)y_{n+2}+(2n+1)x\ y_{n+1}+n^2\ y_n=0.$

Now in this equation let x tend to 0. We get

$$(y_{n+2})_{x=0}=-n^2\ (y_n)_{x=0} \qquad (iii)$$

Also as x tends to 0, we have

$(y)_{x=0}=0$, $(y_1)_{x=0}=0$ from (i) and $(y_2)_{x=0}=2.$

Hence (iii) gives

$(y_3)_{x=0}=(y_5)_{x=0}=\ldots=(y_{2n+1})_{x=0}=0$

and $(y_4)_{x=0}=-2^2.2.$

$(y_6)_{x=0}=4^2.2^2.2$

$(y_8)_{x=0}=-6^2.4^2.2^2.2$

.....

$(y_{2n})_{x=0}=(-1)^{n-1}\ 2.2^2.4^2.6^2\ldots\ldots(2n-2)^2.$

Observe that the process is valid only when the derivatives are all continuous at $x=0$.

9. The Operator D. Let D be a symbol which stands for differentiation, say with respect to x on the function that follows it. That is to say, let

$$Dy=\frac{d}{dx}\ (y) \quad \text{or} \quad Dy=\frac{dy}{dx}.$$

Then $\mathrm{D}(\mathrm{D}y)=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}$

$$\mathrm{D}\{\mathrm{D}(\mathrm{D}y)\}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d^3y}{dx^3}.$$

It follows therefore that the result of the operation which D stands taken n mes in succession will be $\frac{d^ny}{dx^n}$.

Also let the operations which consist of the operation D repeated two, three.....n times in succession be denoted by $\mathrm{D}^2, \mathrm{D}^3, \ldots\ldots \mathrm{D}^n$ so that D^ny stands for $\frac{d^ny}{dx^n}$ and not $\left(\frac{dy}{dx}\right)^n$.

From the above it follows that $\mathrm{D}(ay)=a(\mathrm{D}y)$ provided that a is a constant.

$$\mathrm{D}(u+v)=\frac{d}{dx}(u+v)=\frac{du}{dx}+\frac{dv}{dx}=\mathrm{D}u+\mathrm{D}v$$

$$\mathrm{D}^n(u+v)=\frac{d^n}{dx^n}(u+v)=\frac{d^nu}{dx^n}+\frac{d^nv}{dx^n}=\mathrm{D}^nu+\mathrm{D}^nv$$

$$(\mathrm{D}^m+\mathrm{D}^n)y=\frac{d^my}{dx^m}+\frac{d^ny}{dx^n}=\frac{d^ny}{dx^n}+\frac{d^my}{dx^m}=(\mathrm{D}^n+\mathrm{D}^m)y$$

and $\mathrm{D}^m.\mathrm{D}^ny=\frac{d^m}{dx^m}\left(\frac{d^ny}{dx^n}\right)=\frac{d^{m+n}y}{dx^{m+n}}=\frac{d^n}{dx^n}\left(\frac{d^my}{dx^m}\right)=(\mathrm{D}^n.\mathrm{D}^m)y$

It follows that $\mathrm{D}^m\,\mathrm{D}^ny=\mathrm{D}^{m+n}y$.

If a is a constant, the following equations define the meaning of $\mathrm{D}+a$, $a+\mathrm{D}$, $\mathrm{D}a$ and $a\mathrm{D}$, regarded as operators :

$(\mathrm{D}+a)y=\mathrm{D}y+ay=(a+\mathrm{D})y$, $(\mathrm{D}a)y=\mathrm{D}(ay)=a\mathrm{D}y=(a\mathrm{D})y$.

Hence if a, b are constants,

$$(\mathrm{D}+a)(\mathrm{D}+b)y=(\mathrm{D}+a)(\mathrm{D}y+by)=\mathrm{D}^2y+b\mathrm{D}y+a\mathrm{D}y+aby$$
$$=\{\mathrm{D}^2+(a+b)\mathrm{D}+ab\}y.$$

If $a=b$, then $(\mathrm{D}+a)^2y=(\mathrm{D}^2+2a\mathrm{D}+a^2)y$.

Thus so far as addition, subtraction and multiplication are concerned the operator D combines with itself and with *constant* according to the laws of Algebra.

Ex. 1. Evaluate $(\mathrm{D}^2+a^2)\tan x$.

$$(\mathrm{D}^2+a^2)\tan x=\mathrm{D}^2\tan x+a^2\tan x=\frac{d^2}{dx^2}(\tan x)+a^2\tan x.$$
$$=2\sec^2x\tan x+a^2\tan x.$$

Ex. 2. Evaluate $(D-a)^3 \sin x$.

As a is a constant, therefore $(D-a)^3 = D^3 - 3aD^2 + 3a^2D - a^3$.

Hence $(D-a)^3 \sin x$

$$= (D^3 - 3aD^2 + 3a^2D - a^3) \sin x$$
$$= D^3 \sin x - 3aD^2 \sin x + 3a^2 D \sin x - a^3 \sin x$$
$$= -\cos x + 3a \sin x + 3a^2 \cos x - a^3 \sin x.$$

Ex. 3. Prove that $(D-\tan x)^2(x^2 \sec x) = 2 \sec x$.

Hence $\tan x$ is not a constant, therefore we proceed as follows:

$$(D-\tan x)(x^2 \sec x) = D(x^2 \sec x) - x^2 \sec x \tan x$$
$$= 2x \sec x + x^2 \sec x \tan x - x^2 \sec x \tan x$$
$$= 2x \sec x.$$

$$(D-\tan x)^2 x^2 \sec x = (D-\tan x)\{(D-\tan x)(x^2 \sec x)\}$$
$$= (D-\tan x)(2x \sec x)$$
$$= D(2x \sec x) - 2x \sec x \tan x$$
$$= 2 \sec x + 2x \sec x \tan x - 2x \sec x \tan x$$
$$= 2 \sec x.$$

It should be observed that $(D-\tan x)^2$ does not here stand for $D^2 - 2 \tan x D + \tan^2 x$ because $\tan x$ is not a *constant*.

EXAMPLES XI

Find the nth derivative of:

1. $x^2 e^x$. **2.** xe^{ax}. **3.** $x^3 \cos x$. **4.** $x^2 \log x$.

5. $e^x \log x$. **6.** $x^2 \tan^{-1} x$. **7.** $x\, e^{ax} \sin bx$.

8. If $y = e^{ax}X$, where X is a function of x, show that

$$y_n = e^{ax}\left(X_n + naX_{n-1} + \frac{n(n-1)}{\lfloor 2} a^2 X_{n-2} + \ldots\ldots\right), \text{ where } X_n = \frac{d^n X}{dx^n}$$

9. Find the value of $\frac{d^8}{dx^8}\left(\frac{\sin^3 x \cos x}{256}\right)$.

10. If $y = \frac{x}{a^2+x^2}$, prove that $y_n = (-1)^n \underline{n}\, a^{n-1} \sin^{n+1}\theta \cos(n+1)\theta$

where $\theta = \tan^{-1}\left(\frac{a}{x}\right)$. [*Hint.* $\frac{x}{a^2+x^2} = \frac{1}{2}\left(\frac{1}{x-ai} + \frac{1}{x+ai}\right)$

11. If $y = A(x+\sqrt{x^2-1})^n + B(x-\sqrt{x^2-1})^n$ prove that

$$(x^2-1)y_2 + xy_1 - n^2 y = 0.$$

Differentiate the latter equation n times by Leibnitz's Theorem.

12. If $x=\cos h\left(\frac{1}{m}\log y\right)$ prove that

$$(x^2-1)\,y_2+xy_1-m^2y=0,$$

and $(x^2-1)y_{n+2}+(2n+1)xy_{n+1}+(n^2-m^2)y_n=0.$

13. If $y=\sin(m\sin^{-1}x)$, prove that

$$(1-x^2)y_2=xy_1-m^2y,$$

and $(1-x^2)y_{n+2}=(2n+1)xy_{n+1}+(n^2-m^2)y_n.$

Use the latter equation to find the value of y_n for $x=0.$

14. If $y=e^{a\sin^{-1}x}$, show that

$$(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-(n^2+a^2)y_n=0.$$

15. If $y=(\sin^{-1}x)^2$ show that $(1-x^2)y_2-xy_1-2=0$ and deduce that $(1-x^2)\dfrac{d^{n+2}y}{dx^{n+2}}-(2n+1)x\dfrac{d^{n+1}y}{dx^{n+1}}-x^2\dfrac{d^ny}{dx^n}=3.$

16. If $y=(\sin^{-1}x)^2$; prove that $\underset{x\to0}{\text{Lt}}\left(\dfrac{y_{n+2}}{y_n}\right)=n^2.$

17. If $y=\dfrac{\sin^{-1}x}{\sqrt{1-x^2}}$, prove that for $x=0$, $y_n=(n-1)^2y_{n+2}$

18. If $y=(x+\sqrt{1+x^2})^m$, find $(y_n)x=0.$

19. If $y=a\cos(\log x)+b\sin(\log x)$ show that $x^2y_2+xy_1+y=0$, and $x^2y_{n+2}+(2n+1)xy_{n+1}(n^2+1)y_n=0.$

20. If $y=e^{\tan^{-1}x}$ show that

$(1+x^2)y_2+(2x-1)y_1=0$, $(1+x^2)y_{n+2}+\{2(n+1)x-1\}y_{n+1}+n(n+1)y_n=0.$

21. Show that $(D-\tan x)^3(x^3\sec x).=6\sec x$

22. If $p^2=a^2\cos^2\theta+b^2\sin^2\theta$, prove that $p+\dfrac{d^2p}{db^2}=\dfrac{a^2b^2}{p^2}.$

23. For $y=x^ne^{ax}$, $\dfrac{d^2y}{dx^2}-4\dfrac{dy}{dx}+4y=0$, find n and $a.$

24. If $x+y=1$, show that

$$\frac{d^n}{dx^n}(x^ny^n)=\underline{|n}\{y^n-(nc_1)^2y^{n-1}x+(nc_2)^2y^{n-2}x^2+\ldots+(-1)^nx^n\}$$

25. If $y=\frac{1+x}{1-x}$, prove that

$$D^n y^n=\frac{\lfloor n}{(1-x)^n}\left(1+\frac{n^2}{1^2}\ y+\frac{n^2(n^2-1)}{1^2.2^3}y^2+\ldots\ldots\right)$$

26. Prove that

$$D^n\left(\frac{\log x}{x}\right)=\frac{(-1)^n\ \lfloor n}{x^{n+1}}\left(\log x-1-\frac{1}{2}-\frac{1}{n}\ \ldots\ldots-\frac{1}{n}\right).$$

7. If $y=x^{n-1}\log x$, prove that $x\frac{d^n y}{dx^n}=\underline{\lfloor n-1}$.

28. Show that

$$D^n\left(\frac{\sin x}{x}\right)=\frac{P\sin\left(x+\frac{n\pi}{2}\right)+Q\cos\left(x+\frac{n\pi}{2}\right)}{x^{n+1}}$$

$$\text{and } D^n\left(\frac{\cos x}{x}\right)=\frac{P\cos\left(x+\frac{n\pi}{2}\right)-Q\sin\left(x+\frac{n\pi}{2}\right)}{x^{n+1}}$$

where $P=x^n-n(n-1)x^{n-2}+n(n-1)(n-2)(n-3)x^{n-4}-\ \ldots$ and $Q=nx^{n-1}-(n-1)(n-2)x^{n-3}+\ldots\ldots$

29. If $I_n=x^{n-1}e^{\frac{1}{x}}$, prove by the method of Induction that

$$\frac{d^n I_n}{dx^n}=(-)^n\frac{e^{\frac{1}{x}}}{x^{n+1}}\text{(Halphen)}$$

30. If $y=x^2e^x$. show that

$$\frac{d^n y}{dx^n}=\frac{1}{2}n(n-1)\frac{d^2y}{dx^2}-n(n-2)\ \frac{dy}{dx}+\frac{1}{2}(n-1)(n-2)y$$

31. Prove that if $ac>b^2$

$$\frac{d^n}{dx^n}\left(\frac{b+cx}{a+2bx+cx^2}\right)=(-)^n\ \lfloor n\ \left(\frac{c}{a+2bx+cx^2}\right)^{\frac{n+1}{2}}$$
$$\times\cos\left((n+1)\tan^{-1}\frac{\sqrt{ac-b^2}}{b+cx}.\right)$$

32. Show that the nth derivative of $(e^x-e^{-x})\sin x$ vanishes for the value of $x=0$ unless n is of the form $4m-2$ when its value is $(-1)^{n+1}2^n$

33. Prove that the value when $x=0$ of $\frac{d^n}{dx^n}\tan^{-1}x$ is 0, $\underline{|n-1}$, $-\underline{|n-1}$ according as n is of the form $2m$, $4m+1$, $4m+3$, m being an integer.

34. Find the nth derivative of $\frac{1}{1+x+x^2+x^3}$ and show that for $x=0$, it is 0 when n is of the form $4p+2$ or $4p+3$ and it is $\underline{|n}$ or $-\underline{|n}$ according as n is of the form $4p$ or $4p+1$.

35. Show that the nth derivative of $\sin x \sinh x$ vanishes for $x=0$ unless n be of the form $4m+2$ when its value is

$$(-)^m 2^{2m+1}.$$

36. Prove that the value of nth derivative of $\frac{x^3}{x^2-1}$ for $x=0$ is zero if x is even and is $-\underline{|n}$ if x is odd and greater than 1.

37. From the identity $x^m, x^n = x^{m+n}$ prove by Leibnitz's Theorem that

$$m_r + {}^rC_1 m_{r-1} n_1 + {}^rC_2 m_{r-2} n_2 + \ldots\ldots \ldots + n_r = (m+n)_r, \text{ where}$$
$$m_r = m(m-1)(m-2)\ldots\ldots(m-r+1).$$

38. By finding in different ways the nth derivative of x prove that

$$1+\frac{n^2}{1^2}+\frac{n^2(n-1)^2}{1^2.2^2}+\frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2}-+\ldots\ldots=\frac{\underline{|2}}{(\underline{|n})}.$$

39. Prove by induction or otherwise that if D stands for $\cos^2\theta\frac{d}{d\theta}$,

then $D^n(\frac{1}{2}\sin 2\theta)=(-1)^n \underline{n}(\cos\theta)^{n+1}\cos\left\{(n+1)\left(\frac{\pi}{2}-\theta\right)\right\}$,

CHAPTER IV

PARTIAL DIFFERENTIATION

1. Although in this elementary work we are concerned mainly with functions of a single independent variable, it will occasionally be useful to have an idea of functions of two or more idependent variables, a thorough treatment of such functions is difficult and we will restrict the discussion to simpler properties only.

2. Functions of Several Variables. If the values, the variable u takes, depend on the values, attributed to the variables x, y..... .....then u is said to be a function of these variables and is usually written as $u=f(x, y\ldots\ldots\ldots)$.

If we consider only two independent variables x and y a particular pair of values of x and y can be represented geometrically by a point P in a plane, whose rectangular co-ordinates are x and y. The simultaneous variation of the two variables is then represented by the displacement of the point P in the plain. When we can give to x and y all the systems of values, corresponding to the point P lying anywhere in a portion D of the plane bounded by a closed contour C, and on the contour itself then x and y are said to vary or lie in the domain D, the curve C is called the frontier of the domain. In particular, if x and y vary respectively between (x_1, x_2) and (y_1, y_2) then they are said to lie in the rectangular domain bounded by these values.

The variation of three independent variables x, y and z can also be represented geometrically by the variation of the point P in space, whose rectangular co-ordinates are x, y and z. If the point P can take up all positions in certain volume D bounded by a closed surface S and on the surface itself, then x, y and z are said to vary in the domain D and S is the frontier of the domain. If x, y, z vary respectively between (x_1, x_2), (y_1, y_2) and (z_1, z_2), then the domain is said to be rectangular.

In the case of more than three variables our geometrical intuition fails to provide us with a picture of a region in more than three-dimensional space but, we say by analogy with two three dimensional space that the variables vary in *hyper space* extend the above terminology to the general case. In

particular if x, y, z, t.....vary respectively between (x_1, x_2), (y_1, y_2), (z_1, z_2) (t_1, t_2)..... ..the domain is said to be rectangular.

3. Definition of Limit. Let $f(x, y)$ be defined at all points in the neighbourhood of a point (a, b) except at the point itself (it may be defined at the point, but this fact is quite irrelevent in the definition). Then $f(x, y)$ is said to tend to a limit c when $x \to a$, $y \to b$, if, corresponding to a pre-assigned positive number ϵ, however small, we can find a positive number δ such that

$$| f(x, y) - c | < \epsilon$$

for $| x - a | < \delta$, $| y - b | < \delta$

The definition may be illustrated geometrically as follows:—

Represent the function by a surface $z = f(x, y)$. Draw the planes.

$z = c + \epsilon$, $z = c - \epsilon$ ... ..(*i*)

$x = a + \delta$, $x = a - \delta$, $y = b + \delta$, $y = b - \delta$. .....(*ii*)

These six planes form a rectangular parallelopiped.

To say that $f(x, y)$ tends to c as its limit is merely to say that those points (x, y, z) of the locus $z = f(x, y)$ for which (x, y) lies in the square (*ii*) of the xy-plane—the centre (a, b) being omitted—are all continued within this parallelopiped.

It should not be supposed that

$$\lim_{x \to a} \left\{ \lim_{y \to b} f(x, y) \right\} \text{ is always equal to } \lim_{y \to b} \left\{ \lim_{x \to a} f(x, y) \right\}$$

For example

$$\lim_{x \to 0} \left\{ \lim_{y \to 0} \frac{y - x}{y + x} \right\} = \lim_{x \to 0} \frac{-x}{x} = -1, \text{ whereas}$$

$$\lim_{y \to 0} \left\{ \lim_{x \to 0} \frac{y - x}{y + x} \right\} = \lim_{y \to 0} \frac{y}{y} = 1.$$ But usually these two limits are equal. This study of double limits is very stiff and beyond the scope of this book.

The function $f(x, y)$ is said to be *continuous at the point* (a, b) if $f(x, y)$ is defined at every point in the neighbourhood of (a, b) and has for limit $f(a, b)$ when x and y tend respectively

towards a and b in any manner whatsoever, **that is to say, given** ϵ, a $\delta(\epsilon)$ can be found such that

$$| f(x, y)-f(a, b) | <\epsilon$$

whenever $| x-a | <\delta$ and $| y-b | <\delta$.

If (a, b) lies on the boundary of the region, only such points (x, y) are considered as lying in the domain.

This definition can be easily extended to a function of any number of variables.

4. Partial Derivatives. Let $z=f(x, y)$ be a continuous function of two independent variables x and y. If we give to y a constant value and let x vary, z becomes a continuous function of x alone ; if it possesses a derivative, that derivative is called *partial derivative of z with respect to x.* This partial derivative is, therefore, by definition, the limit of the ratio

$$\frac{f(x+\delta x, y)-f(x, y)}{\delta x} \text{ when } \delta x \to 0$$

and is usually denoted by f_x, $f'_x(x, y)$, $D_x f(x, y)$, $\frac{\partial f'(x, y)}{\partial x}$, $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ to distinguish it from the differential co-efficient $\frac{df}{dx}$ obtained by making the remaining variable y an arbitrary functin of x and differentiating by the *theorem on differentiation of function of function.*

Similarly, regarding x as a constant and y as the variable, we define the **partial derivative of z with respect to y** as the limit of the ratio.

$$\frac{f(x, y+\delta y)-f(x, y)}{\delta y} \text{ when } \delta y \to 0.$$

We represent this by the symbols f_y, $f'_y(x, y)$, $D_y f(x, y)$ $\frac{\partial f(x, y)}{\partial y}$, $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$.

5. Partial Differentials. $\frac{\partial f}{\partial x}\delta x$ and $\frac{\partial f}{\partial y}\delta y$ are defined as partial differentials, where δx and δy are arbitrary increments of x and y respectively.

It may be remarked that $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ *are symbolic expresions and cannot be decomposed as quotient of two differentials.*

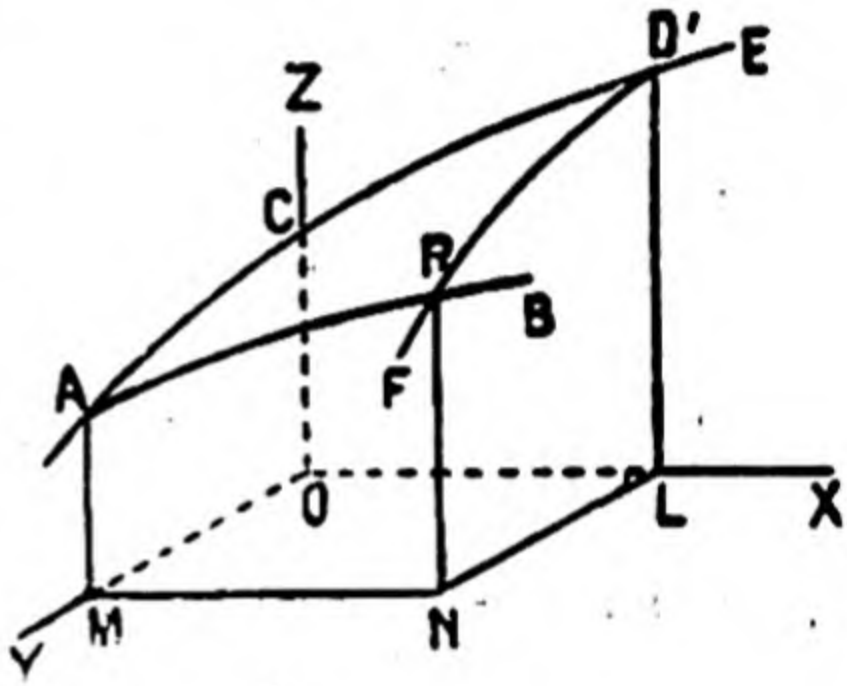
Geometrical interpretation.

The partial derivatives $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ can be easily interpreted geometrically. Let P be the point (x, y, z) on the surface given by $z=f(x, y)$ and let DPF and APB be sections made by planes through P parallel to the planes YOZ and ZOX respectively.

For points on the curve APB, y is constant.

Hence $\frac{\partial z}{\partial x}$ *is the gradient at P of the curve APB*, that is, the tangent of the angle which the tangent at P to the curve APB makes with the x-axis.

Similarly $\frac{\partial z}{\partial x}$ *is the gradient at P of the curve DPF.*



Ex. 1. Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$ when $z=x^3+y^3-3axy$.

Here $\frac{\partial z}{\partial x}=3x^2-3ay$ and $\frac{\partial z}{\partial y}=3y^2-3ax$.

Ex. 2. If $f(x, y)=\sin(e^{ax}+e^{by})$, find $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$.

Here $\frac{\partial f}{\partial x}=ae^{ax}\cos(e^{ax}+e^{by})$

and $\frac{\partial f}{\partial y}=be^{by}\cos(e^{ax}+e^{by})$.

Ex. 3. If $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$$

$$u=\tan^{-1}\frac{x^3+y^3}{x-y}$$

$$\therefore x\frac{\partial u}{\partial x}=x.\ \frac{1}{1+\left(\frac{x^3+y^3}{x-y}\right)^2}\times\frac{(x-y)\times 3x^2-(x^3+y^3)\times -1}{(x-y)^2}$$

$$=\frac{x\times(2x^3-3x^2y-y^3)}{(x-y)^2+(x^3+y^3)^2}$$

$$y\frac{\partial u}{\partial y}=y\times\frac{1}{1+\left(\frac{x^3+y^3}{x-y}\right)^2}\times\frac{(x-y)\ 3y^2-(x^3+y^3)\times -1}{(x-y)^2}$$

$$=\frac{y\times(3y^2x-2y^3+x^3)}{(x-y)^2+(x^3+y^3)^2}.$$

$$\therefore x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2(x^4-y^4-x^3y+xy^3)}{(x-y)^2+(x^3+y^3)^2}$$

$$=\frac{2(x^3+y^3)(x-y)}{(x-y)^2+(x^3+y^3)^2}.$$

Again, $\sin 2u=\dfrac{2\tan u}{1+\tan^2 u}=\dfrac{2.\dfrac{x^3+y^3}{x-y}}{1+\left(\dfrac{x^3+y^3}{x-y}\right)^2}$

$$=\frac{2(x^3+y^3)(x-y)}{(x-y)^2+(x^3+y^3)^2}.$$

Hence the result.

6. Partial derivatives of the second order may be defined in a similar manner. The partial derivatives $f'_x(x, y)$ and $f'_y(x, y)$ are themselves functions of the two variables x and y and might therefore possess partial derivatives. The derivative of $f_x(x, y)$ with respect to x is denoted by f_{xx}, $f''_{xx}(x, y)$ or $\dfrac{\partial^2 f}{\partial x^2}$, and that with respect to y is denoted by the symbols of f_{xy}, $f''_{xy}(x, y)$, or $\dfrac{\partial^2 f}{\partial y\partial x}$.

In the same way, the partial derivatives of $f'_y(x, y)$ with respect to x and y are respectively denoted by the symbol as f_{yx}, f''_{yx} or $\dfrac{\partial^2 f}{\partial x\partial y}$ and f_{yy}, $f''_{yy}f(x, y)$, or $\dfrac{\partial^2 f}{\partial y^2}$.

Similarly, **partial derivatives of higher orders** can be easily defined and are denoted by corresponding symbols.

Definitions, similar to those given above, hold in the case of three or more independent variables.

Ex. 1. If $f(x, y)=x^m y^n$, find all the partial derivatives up to the second order and show that $\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial^2 f}{\partial y \partial x}$ in this case.

Here $\frac{\partial f}{\partial x}=mx^{m-1} y^n$, $\frac{\partial^2 f}{\partial x^2}=m(m-1)x^{m-2}y^n$

$$\frac{\partial^2 f}{\partial y \partial x}=mnx^{m-1}y^{n-1}$$

and $\frac{\partial f}{\partial y}=xy^{n-1} x^m$, $\frac{\partial^2 f}{\partial y^2}=n(n-1)y^{n-2}x^m$.

$$\frac{\partial^2 f}{\partial x \partial y}=nmy^{n-1} x^{m-1}.$$

so that $\frac{\partial^2 f}{\partial x \partial y}=\frac{\partial^2 f}{\partial x \partial y}$ in this case.

Ex. 2. Find $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ in the following cases and show that they are equal :—(*i*) $u=e^{my} \sin nx$

(*ii*) $\log (e^x+e^y)$.

(*i*) $\frac{\partial u}{\partial x}=e^{my}n \cos nx$, $\frac{\partial^2 u}{\partial y \partial x}=nm\, e^{my} \cos nx$

and $\frac{\partial u}{\partial y}=m\, e^{my} \sin nx$, $\frac{\partial^2 u}{\partial x \partial y}=mn\, e^{my} \cos nx$

therefore $\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}$ in this case.

(*ii*) $\frac{\partial u}{\partial x}=\frac{e^x}{e^x+e^y}$. $\frac{\partial^2 u}{\partial y \partial x}=-\frac{e^x e^y}{(e^x+e^y)^2}$

$\frac{\partial u}{\partial y}=\frac{e^y}{e^x+e^y}$, $\frac{\partial u}{\partial x \partial y}=-\frac{e^y . e^x}{(e^x+e^y)^2}$.

Hence $\frac{\partial^2 u}{\partial x \partial y}=\frac{\partial^2 u}{\partial y \partial x}$ in this case also.

In fact it can be proved that under certain conditions **the order in which the successive differentiations are performed is immaterial.**

7. Theorem.* *If a continuous function* $u=f(x, y)$ *possesses first and second partial derivatives continuous in the neighbourhood of a point* (x, y), *then at that point*

$$\frac{\partial^2 u}{\partial y \partial x}=\frac{\partial^2 u}{\partial x \partial y}.$$

$\frac{\partial^2 f}{\partial y \partial x}$ and $\frac{\partial^2 f}{\partial x \partial y}$ are defined by the equations :

$$\frac{\partial^2 f}{\partial y \partial x}=\operatorname*{Lt}_{\delta y \to 0}\ \operatorname*{Lt}_{\delta x \to 0}\left\{\frac{f(x+\delta x, y+\delta y)-f(x, y+\delta y)}{\delta x}-\frac{f(x+\delta x, y)-f(x, y)}{\delta x}\right\}/\delta y.$$

$$\frac{\partial^2 f}{\partial x \partial y}=\operatorname*{Lt}_{\delta x \to 0}\ \operatorname*{Lt}_{\delta y \to 0}\left\{\frac{f(x+\delta x, y+\delta y)-f(x+\delta x, y)}{\delta y}-\frac{f(x, y+\delta y)-f(x, y)}{\delta y}\right\}/\delta x.$$

The expressions on the right are identically equal to

$$\{f(x+\delta x, y+\delta y)-f(x, y+\delta y)-f(x+\delta x, y)+f(x, y)\}/\delta x \delta y \equiv \psi(\delta x, \delta y).$$

Applying the Mean Value Theorem to the function

$$\phi(x, y)=f(x, y+\delta y)-f(x, y)$$

in which y and δy are regarded as constant and x variable, we have

$$\phi(x+\delta x, y)-\phi(x, y)=\delta x \frac{\partial}{\partial x}\phi(x+\theta_1 \delta x, y)$$

$$=\delta x\left(\frac{\partial}{\partial x} f(x+\theta_1 \delta x, y+\delta y)-\frac{\partial}{\partial x} f(x+\theta_1 \delta x, y)\right)$$

(Apply the Mean Value Theorem to this difference also)

$$=\delta y\, \delta x \frac{\partial}{\partial y}\frac{\partial}{\partial x} f(x+\theta_1 \delta x, y+\theta_2 \delta y).$$

where θ_1, θ_2 are positive proper fractions.

$$\therefore \quad \psi(\delta x, \delta y)=\frac{\partial^2}{\partial y\, \partial x} f(x+\theta_1 \delta x, y+\theta_2 \delta y).$$

Similarly $\psi(\delta x, \delta y) = \dfrac{\partial^2}{\partial x \partial y} f(x+\theta_3 \delta x, y+\theta_4 \delta y)$.

Let δx and δy tend to zero in any order, ψ represents the same value independently of the order.

$$\therefore \qquad \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

The theorem holds under more general conditions, and is beyond the scope of this book.

EXAMPLES XII

1. Find the first partial derivatives of

$$u = \frac{1}{\sqrt{x^2+y^2}}.$$

Find $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial^2 u}{\partial y^2}$ in the following :

2. $u = \tan^{-1}\dfrac{x}{y}$. **3.** $u = e^{m(x+y)}$. **4.** $u = e^y \sin x$.

Show that in the following cases $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$:—

5. $f(x, y) = \dfrac{1}{\sqrt{x+y}}$. **6.** $f(x, y) = e^{mx} \cos ny$.

7. $f(x, y) = \sin(e^x + e^y)$. **8.** $f(x, y) = \sin^{-1}\dfrac{x}{y}$.

9. If $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}}$, verify that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = -u.$$

10. If $u = ax^2 + by^2 + cz^2$, verify that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2u.$$

11. Find $\dfrac{\partial u}{\partial x}$, $\dfrac{\partial^2 u}{\partial x^2}$, $\dfrac{\partial^2 u}{\partial x \partial y}$ where

$$u=x^2\tan^{-1}\frac{y}{x}-y^2\tan^{-1}\frac{x}{y},$$

it being assumed that x and y are both different from zero.

12. If $u=(x^2+y^2+z^2)^{-\frac{1}{2}}$, show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=0.$$

13. If $u=\sin^{-1}\frac{x}{y}+\tan^{-1}\frac{y}{x}$ show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=0.$$

14. If $\theta=t^n e^{-\frac{r^2}{4t}}$ find n such that

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)=\frac{\partial\theta}{\partial t}.$$

15. If $z=(\alpha x^2+\beta y^2)^{\frac{1}{2}}$ prove that

$$\alpha\frac{\partial^2 z}{\partial y^2}+\beta\frac{\partial^2 z}{\partial x^2}=\frac{\alpha\beta}{z}.$$

16. If $u=f(x^2+y^2)$, prove that

$$\frac{\partial u}{\partial x}:\frac{\partial u}{\partial y}=x:y.$$

17. Verify that if $z=3xy-y^3+(y^2-2x)^{\frac{3}{2}}$,

then $\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial^2 z}{\partial y\,\partial x}$ and $\frac{\partial^2 z}{\partial x^2}\cdot\frac{\partial^2 z}{\partial y^2}=\left(\frac{\partial^2 z}{\partial x\,\partial y}\right)^2$,

18. Show that if $z=x\phi\left(\frac{y}{x}\right)+\psi\left(\frac{y}{x}\right)$

then $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$

19. If $u=\sin^{-1}\sqrt{\frac{x-y}{y}}$, show that

$$\frac{\partial^2 u}{\partial x\,\partial y}=\frac{1}{4}\frac{y}{(xy-y^2)^{\frac{3}{2}}}=\frac{\partial^2 u}{\partial y\,\partial x}.$$

8. Differentiation of Composite Functions.

Let $u=f(x, y)$ be a function of the variables x and y; possessing continuous partial derivatives, f_x, f_y:

Further let $x=\phi(t)$, $y=\psi(t)$ be single valued functions of t such that $\phi'(t)$ and $\psi'(t)$ exist and are continuous.

Then u is a composite function of t which possesses a derivative given by the equation

$$\frac{du}{dt}=\frac{\partial f}{\partial x}\frac{dx}{dt}+\frac{\partial f}{\partial y}\cdot\frac{dy}{dt}$$

that is
$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt} \qquad \text{.....(A)}$$

For, by hypothesis, to each value of t corresponds a unique point (x, y) and to this point (x, y) a value of u. Hence u may be regarded as a function of t.

When t takes an increment δt, let x, y and consequently u take the corresponding increments δx, δy, and δu.

Then $\delta u=f(x+\delta x, y+\delta y)-f(x, y)$

$\therefore\ \delta u=[f(x+\delta x, y+\delta y)-f(x, y+\delta y)]$
$+[f(x, y+\delta y)-f(x, y)]$

We apply Mean Value Theorem to these differences.

The first difference

$f(x+\delta x, y+\delta y)-f(x, y+\delta y)=\delta x f_x(x+\theta_1\delta x, y+\delta y)$

[regarding $f(x, y+\delta y)$ as a function of x alone and letting x range in the interval $(x, x+\delta)x$]

and the 2nd difference

$$f(x, y+\delta y)-f(x, y)=\delta y f_y(x, y+\theta_2\delta y)$$

[regarding $f(x, y)$ as a function of y alone in the interval $(y, y+\delta y)$], where θ_1 and θ_2 are positive proper fractions. Hence $\delta u=fx(x+\theta_1\,\delta x, y+\delta y)\,\delta x+f_y(x, y+\theta_2\,\delta y)\,\delta y$...(1).

Now we may write

$$f_x(x+\theta_1\delta x, y+\delta y)=f_x(x, y)+\alpha$$

and $f_y(x, y+\theta_2\delta y)=f_y(x, y)+\beta$

where Lt $\alpha=0$, Lt $\beta=0$ when δx and δy both tend to zero.

Thus $\delta u=\dfrac{\partial f}{\partial x}\delta x+\dfrac{\partial^f}{\partial y}\delta y+\alpha\delta x+\beta\delta y$.....(2)

Dividing throughout by δt and letting $\delta t\to 0$ as its limit we have

$$\underset{\delta t\to 0}{\text{Lt}}\ \frac{\delta u}{\delta t}=\underset{\delta t\to 0}{\text{Lt}}\ \frac{\partial f}{\partial x}\frac{\delta x}{\delta t}+\underset{\delta t\to 0}{\text{Lt}}\ \frac{\partial f}{\partial y}\frac{\delta y}{\delta t}$$

But $\text{Lt} = \frac{dx}{dt}$, $\text{Lt}\ \frac{\delta y}{\delta t} = \frac{dy}{dt}$.

Thus $$\frac{du}{dt} = \frac{\partial f}{\partial x}\ \frac{dx}{dt} + \frac{\partial f}{\partial y}\ \frac{dy}{dt}$$

or $$\frac{du}{dt} = \frac{\partial u}{\partial x}\ \frac{dx}{dt} + \frac{\partial u}{\partial y}\ \frac{dy}{dt}. \qquad \ldots(A)$$

Note 1. In the same way it can be shown that if $u=f(x, y, z)$ where x, y, z are function of t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\ \frac{dx}{dt} + \frac{\partial u}{dy}\ \frac{dy}{dt} + \frac{\partial u}{\partial z}\ \frac{dz}{dt}.$$

This can be generalised for any number of variables x, y, z ..... all functions of t.

Note 2. In the above work, if we assume that $\phi(t)=t$, that is $x=t$, then $u=f(t, y)$ and equation (A) becomes

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial y}\ \frac{dy}{dt}.$$

Here $\frac{du}{dt}$ and $\frac{\partial u}{\partial t}$ have different meanings.

The derivative $\frac{\partial u}{\partial t}$ is formed on the supposition that *an explicitly named* variable t alone varies ; on the other hand $\frac{du}{dt}$ is the limit of $\frac{\delta u}{\delta t}$ where δu is the change in u due (*i*) to the change δt in the *explicitly named* variable t and (*ii*) the change δy which is itself due to the change δt.

Note. 3. If $u=f(x, y, z.....)$ where $x=\phi(t, r...)$ $y=\psi(t, r...)$ then it follows from (A) that

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial u}{\partial x}\cdot\ \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\cdot\ \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z}\cdot\ \frac{\partial z}{\partial t} + \ldots\ldots \\ \frac{\partial u}{\partial r} &= \frac{\partial u}{\partial x}\cdot\ \frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\ \frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\cdot\ \frac{\partial z}{\partial r} + \ldots\ldots \end{aligned}\right\} \quad \ldots(B)$$

and so on for other variables s etc.

Ex. 1. If $u=x^2+y^2$, then $\frac{\partial u}{\partial x}=2x$ and $\frac{\partial u}{\partial y}=2y$; if

$x=at^2$ and $y=2at$, then

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt}$$
$$=2x.2at+3y.2a$$
$$=4a^2t^3+8a^2t.$$

If we directly take $u=a^2t^4+4a^2t^2$, we again get

$$\frac{du}{dt}=4a^2t^3+8a^2t.$$

Ex. 2. If $u=f(t, y)=t^2+y^3$, where y itself is a function of t, say $y=t^2+1$, find by the above rule $\frac{du}{dt}$ and verify the result.

Here $$\frac{du}{dt}=\frac{\partial u}{\partial t}+\frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$
$$=2t+2y\,2t$$
$$=2t+4t(t^2+1)$$
$$=4t^3+6t.$$

Also directly $u=t^2+(t^2+1)^2=t^4+3t^2+1$

and $$\frac{du}{dt}=4t^3+6t.$$

9. Differentiable Functions and Total Differentials.

Let $u=f(x, y)$ be a function of two variables and let δu be the increment in u corresponding to the arbitrary increments δx and δy given to x and y. $u=f(x, y)$ is said to be **differentiable** if we can write

$$\delta u=A\delta x+B\delta y+\alpha\delta x+\beta\delta y \qquad (1)$$

where A and B are independent of δx and δy and α and β tend to zero as δx and δy tend to zero in any manner whatsoever. As particular cases

A, B, α, β may be zero

The part $A\delta x+B\delta y$ is defined as the **total differential** of u and is denoted by du.

If we take $\delta y=0$ and then let δx tend to zero, we get

$$\lim_{\delta x\to 0}\frac{\delta u}{\delta x}=\frac{\partial u}{\partial x}=A.$$

Similarly $$\frac{\partial u}{\partial y}=B.$$

Therefore $$du=\frac{\partial u}{\partial x}\delta x+\frac{\partial u}{\partial y}\delta y \qquad \ldots(2)$$

If we take $u=x$ or $u=y$ in (2), we get

$$dx=\delta x \text{ and } dy=\delta y,$$

and therefore replacing δx and δy by dx and dy in (2) we get

$$d\cdot=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy \qquad \ldots(3)$$

It may be remarked that equation (3) is more general than equation (2) because in (2), x and y are independent variables whereas equation (3) is free from such restrictions.

Thus we have

$$du=f_x dx+f_y dy$$

$$\text{or } du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy..... \qquad (4)$$

Equation (3) may also be symbolically written as

$$d\cdot=\left(\frac{\partial}{\partial x}dx+\frac{\partial}{\partial y}\right)f.$$

Similarly for three (or more) independent variables

$$du=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial y}{\partial z}dz.$$

The reader is reminded that when $u=f(x)$ is a function of one variable, then if the derivative exists, the function is differentiable, and if the function is differentiable, the derivative exists. But when $u=f(x, y)$ is a function of two variables, then if the function is differentiable, the partial derivatives exist but the converse is not necessarily true; that is to say, a function may posses partial derivatives but may not be differentiable. However in Article 8 equation (2) above we have incidentally shown that if $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist and are continuous then $f(x, y)$ is differentiable.

10. Successive Total Differentials.

The differential of du is $d(du)$ denoted by d^2u and is determined by applying the above rule.

$$d^2u=d(du)=\frac{\partial(du)}{\partial x}dx+\frac{\partial(du)}{\partial y}dy$$

$$= \left[\frac{\partial^2 f}{\partial x^2} dx + \frac{\partial^2 f}{\partial y \partial x} dy \right] dx + \left[\frac{\partial^2 f}{\partial x \partial y} dx + \frac{\partial^2 f}{\partial y^2} dy \right] dy$$

$$= \frac{\partial^2 f}{\partial x^2} dx^2 + 2 \frac{\partial^2 f}{\partial x \partial y} dx dy + \frac{\partial^2 f}{\partial^2 y} dy^2,$$

which may be symbolically written as

$$d^2u = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} \right)^2 f.$$

More generally, it can be shown that with the same symbolism

$$d^n u = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^n f$$

11. Differentiation of Implicit Functions.

We shall now show how to find out $\frac{dy}{dx}$ when $f(x, y)=0$.

Let $f(x, y)=0$ where $\frac{\partial f}{\partial x}$ and $\frac{\partial}{\partial y}$ exist and are continuous and $\frac{\partial f}{\partial y}$ is not zero.

Since $f(x, y)=0$, therefore $f(x+\delta x, y+\delta y)=0$.

Thus $0 = f(x+\delta x, y+\delta y) - f(x, y)$

$$= [f(x+\delta x, y+\delta y) - f(x, y+\delta y)] + [f(x, y+\delta y) - f(x, y)]$$

Applying the Mean Value Theorem to each difference, we get

$$f_x(x+\theta_1 \delta x, y+\delta y)\, \delta x + f_y(x, y+\theta_2 \delta y)\, \delta y = 0,$$

where θ_1 and θ_2 are positive proper fractions.

Thus $\frac{\delta y}{\delta x} = -\frac{f_x(x+\theta_1 \delta x, y+\delta y)}{f_y(x, y+\theta_2 \delta y)}$.

Proceeding to the limit we have

$$\frac{dy}{dx} = -\frac{f_x}{f_y} \qquad \ldots\ldots(1)$$

This is justified only when $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are continuous and $\frac{\partial f}{\partial y}$ is not zero.

Thus under the conditions stated above if $f(x, y)=0$, then,

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0.$$

Differentiating this again with respect to x by the same rule, we get

$$\frac{\partial}{\partial x}\left[\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}\right]+\frac{\partial}{\partial y}\left[\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}\right]\frac{dy}{dx}=0$$

or
$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y\,\partial x}\frac{dy}{dx}+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}$$
$$+\left(\frac{\partial^2 f}{\partial x\,\partial y}+\frac{\partial^2 f}{\partial y^2}\frac{dy}{dx}\right)\frac{dy}{dx}=0$$

or
$$\frac{\partial^2 f}{\partial x^2}+\frac{2\partial^2 f}{\partial x\,\partial y}\frac{dy}{dx}+\frac{\partial^2 f}{\partial y^2}\left(\frac{dy}{dx}\right)^2+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}=0.$$

We thus get, after $\frac{dy}{dx}$ has been replaced by its value,

$$\frac{d^2y}{dx^2}=-\frac{\frac{\partial^2 f}{\partial x^2}\left(\frac{\partial f}{\partial y}\right)^2-2\frac{\partial f}{\partial x}\frac{\partial f}{\partial y}\frac{\partial^2 f}{\partial x\,\partial y}+\frac{\partial^2 f}{\partial y^2}\left(\frac{\partial f}{\partial x}\right)^2}{\left(\frac{\partial f}{\partial y}\right)^3}.$$

Similarly the higher derivatives can be found.

Ex. If $f(x, y)=x^5+y^5-5a^3xy=0$, find $\frac{dy}{dx}$.

Here $\frac{\partial f}{\partial x}=5(x^4-a^3y)$, $\frac{\partial f}{\partial y}=5(y^4-a^3x)$

$$\therefore \quad \frac{dy}{dx}=-\frac{x^4-a^3y}{y^4-a^3x}.$$

EXAMPLES XIII

1. Given that $f(x, y)=x^2+y^2-a^2=0$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ and verify the result by finding y in terms of x and then differentiating.

2. Given that $f(x, y)=x^3+y^3-3axy=0$, find

$$\frac{dy}{dx} \text{ and } \frac{d^2y}{dx^2}.$$

3. Find the value of $\frac{dB}{dA}$ for a triangle whose angles A, B, C are such that sin B sin C+sin C sin A+sin A sin B=h.

4. Assuming the formula for total differentiation

$$\frac{du}{dt}=\frac{\partial u}{\partial x}\frac{dx}{dt}+\frac{\partial u}{\partial y}\frac{dy}{dt},$$ find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in terms of x, y when $f(x, y)=0$ and prove that

$$\frac{d^2y}{dx^2}\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{-\frac{3}{2}}=\pm\left[f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2\right](f_x^2+f_y^2)^{-\frac{3}{2}}$$

5. If x, y, z are connected by two equations of the form $f(x, y, z)=0$, $\phi(x, y, z)=0$, determine the expression for $\frac{dy}{dx}$.

6. If $x^3+y^2-3z+a=0$, $z^2-2y^2-x+b=0$; find $\frac{dy}{dx}$ and $\frac{dz}{dx}$.

7. Given the equations

$$x^3+y^3+z^3=3xyz \text{ and } x+y+z=0,$$

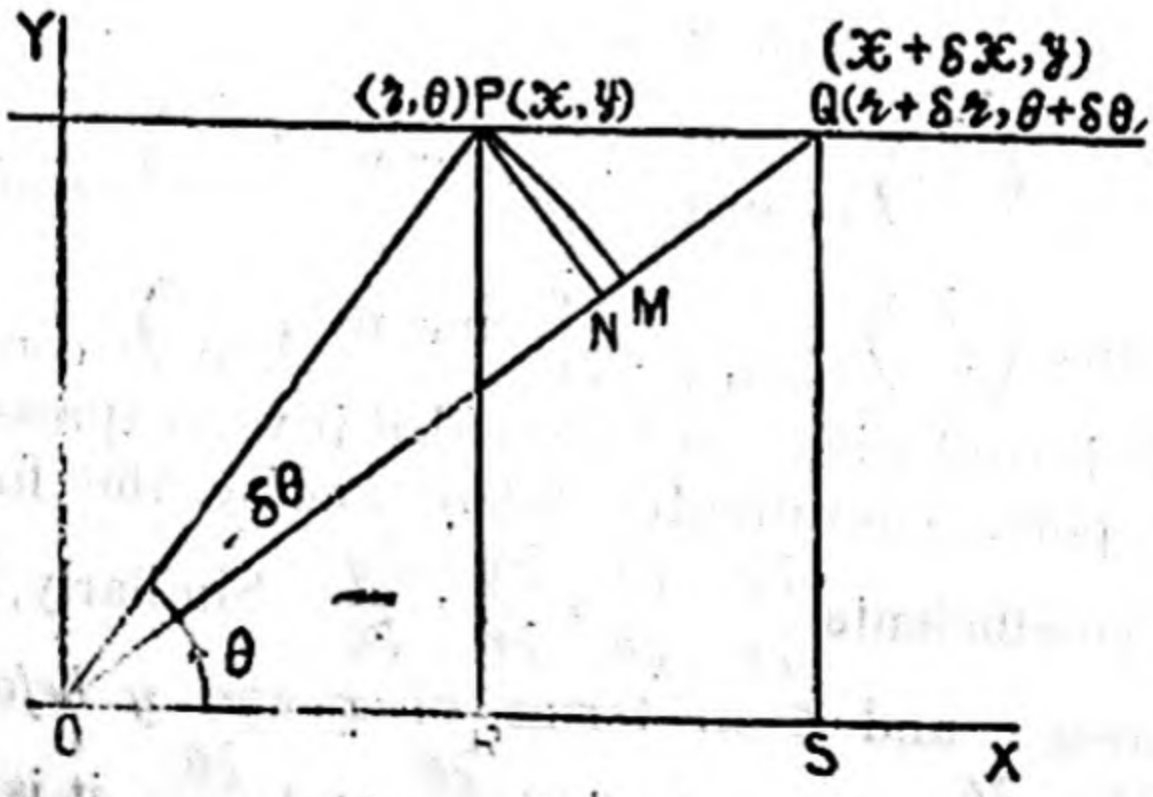
find $\frac{dy}{dx}$ and $\frac{dz}{dx}$.

8. If $x=r\cos\theta$, $y=r\sin\theta$, find the values of

(*i*) $\left(\frac{\partial x}{\partial\theta}\right)_{r \text{ constant}}$ (*ii*) $\left(\frac{\partial\theta}{\partial x}\right)_{y \text{ constant}}$ (*iii*) $\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}$.

Verify the second result geometrically.

[*Hint.*—



PM=$r\,\delta\,\theta$=PN, neglecting infinitesimals of higher order.

$$\frac{PN}{PQ}=\sin \angle NQP \text{ or } -\frac{r\,\delta\theta}{\delta x}=\sin(\theta+\delta\theta).$$

Therefore in the limiting case

$$\frac{\partial\theta}{\partial x}=-\frac{1}{r}\sin\theta=-\frac{y}{x^2+y^2}\Big].$$

9. If $x=r\cos\theta$, $y=r\sin\theta$, show that

(*i*) $\left(\frac{\partial x}{\partial r}\right)_{\theta \text{ constant}}=\left(\frac{\partial r}{\partial x}\right)_{y \text{ constant}}=\cos\theta.$

(*ii*) $\left(\frac{\partial x}{\partial r\theta}\right)_{r \text{ constaut}}=\left(\frac{r\,\partial\theta}{\partial x}\right)_{y \text{ constant}}=-\sin\theta.$

(*iii*) $\frac{\partial r}{\partial x^2}+\frac{\partial^2 r}{\partial y^2}=\frac{1}{r}\left[\left(\frac{\partial r}{\partial x}\right)^2+\left(\frac{\partial r}{\partial y}\right)^2\right].$

Observations on Examples 8 and 9.

1. It is often necessary to change from cartesian to polar co-ordinates or *vice versa*. In this operation we use the equations

$$x=r\cos\theta,\ y=r\sin\theta. \quad \ldots(1)$$

$$\left(\text{Or their equivalents } r=\sqrt{x^2+y^2} \text{ and } \theta=\tan^{-1}\frac{y}{x}\right)$$

These four variables x, y, r, θ are connected by two relations (1), so that any one of them can be expressed in terms of two others. Thus y may be expressed in terms of

(*i*) r, θ or (*ii*) r, x or (*iii*) θ, x. Similar remarks apply to the other variables.

2. In such transformations:

$\frac{\partial}{\partial x}$ means $\left(\frac{\partial}{\partial x}\right)_{y \text{ const.}}$; $\frac{\partial}{\partial y}$ means $\left(\frac{\partial}{\partial y}\right)_{x \text{ constant}}$,

$\frac{\partial}{\partial r}$ means $\left(\frac{\partial}{\partial r}\right)_{\theta \text{ const.}}$; $\frac{\partial}{\partial\theta}$ means $\left(\frac{\partial}{\partial\theta}\right)_{r \text{ constant}}$.

3. Much confusion can be avoided if we express x and y in terms of polar co-ordinates *before finding* the four partial differential co-efficients $\frac{\partial x}{\partial r}$, $\frac{\partial x}{\partial\theta}$, $\frac{\partial y}{\partial r}$, $\frac{\partial y}{\partial\theta}$. Similarly, it is desirable to express r and θ in terms of x and y *before finding* $\frac{\partial r}{\partial x}$, $\frac{\partial r}{\partial y}$, $\frac{\partial\theta}{\partial x}$, $\frac{\partial\theta}{\partial y}$. Thus, to find $\frac{\partial\theta}{\partial x}$ and $\frac{\partial\theta}{\partial y}$, it is more con-

venient to use the relation $\theta=\tan^{-1}\frac{y}{x}$ whence we easily get

$$\left(\frac{\partial\theta}{\partial x}\right)_{y \text{ const.}}=\frac{1}{1+\frac{y^2}{x^2}}\times-\frac{y}{x^2}=-\frac{y}{x^2+y^2}$$

We can also get $\left(\frac{\partial\theta}{\partial x}\right)_y$ by differentiating $y=r\sin\theta$ keeping y constant, but it is very inconvenient as is shown below :

$$0=\left(\frac{\partial r}{\partial x}\right)_y\sin\theta+r\cos\theta\left(\frac{\partial\theta}{\partial x}\right)_y, \text{ whence}$$

$$\left(\frac{\partial\theta}{\partial x}\right)_y=-\left(\frac{\partial r}{\partial x}\right)_y\times\frac{\sin\theta}{r\cos\theta}=\frac{x}{r}\times\frac{\tan\theta}{r}=-\frac{y}{x^2+y^2}.$$

10. If $u=F(x-y, y-z, z-x)$, find the value of

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}.$$

[Put $x-y=t_1$, $y-z=t_2$ and $z-x=t_3$

$$\frac{\partial u}{\partial x}=\frac{\partial u}{\partial t_1}\frac{\partial t_1}{\partial x}+\frac{\partial u}{\partial t_2}\frac{\partial t_2}{\partial x}+\frac{\partial u}{\partial t_3}\frac{\partial t_3}{\partial x}$$

$$=\frac{\partial u}{\partial t_1}\times1+\frac{\partial u}{\partial t_2}\times0+\frac{\partial u}{\partial t_3}\times-1$$

$$=\frac{\partial u}{\partial t_3}+\frac{\partial u}{\partial t_1} \qquad \ldots\ldots(1)$$

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial t_1}\times\frac{\partial t_1}{\partial y}+\frac{\partial u}{\partial t_2}\times\frac{\partial t_2}{\partial y}+\frac{\partial u}{\partial t_3}\times\frac{\partial t_3}{dy}$$

$$=\frac{\partial u}{\partial t_1}\times-1+\frac{du}{\partial t_2}\times1+\frac{du}{\partial t_3}\times0$$

$$=-\frac{\partial u}{\partial t_1}+\frac{\partial u}{\partial t_2}$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial t_1}\times\frac{\partial_1}{\partial z}+\frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial z}+\frac{\partial u}{\partial t_3}\times\frac{\partial t_3}{\partial z} \qquad \ldots\ldots(2)$$

$$=\frac{\partial u}{\partial t_1}\times0+\frac{\partial u}{\partial t_2}\times-1+\frac{\partial u}{\partial t_3}\times1$$

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$$=-\frac{\partial u}{\partial t_2}+\frac{\partial u}{\partial t_3} \qquad \ldots\ldots(3)$$

Adding (1), (2) and (3), we get $\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}=0$.]

11. If $z=f(x-ct)+\phi(x+ct)$, show that $\frac{\partial^2 z}{\partial t^2}=c^2\frac{\partial^2 z}{\partial x^2}$.

12. If $u=x^2+y^2$, show (without assuming any formula) that

$$du=2x\,dx+2y\,dy.$$

13. If $u=\tan^{-1}\frac{y}{x}$, prove that $du=(x\,dy-y\,dx)\div(x^2+y^2)$.

14. If $pv=k\theta$ (k constant), show that $dp=-\frac{p}{v}\,dv+\frac{p}{\theta}\,d\theta$.

15. If $x=r\cos\theta$, $y=r\sin\theta$, r and θ being independent, show that

$$dx=\cos\theta\,dr-r\sin\theta\,d\theta\ ;\ dy=\sin\theta\,dr+r\cos\theta\,d\theta$$

and $x\,dy-y\,dx=r^2d\theta$.

Also prove that $d^2x=\cos\theta d^2r-2\sin\theta dv\,d\theta-r\cos\theta\,d\theta^2$.
and $d^2y=\sin\theta\,d^2r+2\cos\theta dr\,d\theta-r\sin\theta\,d\theta^2$.

16. Given $x=r\cos\theta$, $y=r\sin\theta$, where r and θ are functions of t, prove that

$$\frac{dx}{dt}=\frac{dr}{dt}\cos\theta-r\sin\theta\frac{d\theta}{dt}$$

$$\frac{dy}{dt}=\frac{dr}{dt}\sin\theta+r\cos\theta\frac{d\theta}{dt}$$

$$\frac{d^2x}{dt^2}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta^2}{dt}\right)^2\right]\cos\theta-\left(2\frac{dr}{dt}\cdot\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}\right)\sin\theta$$

$$\frac{d^2y}{dt^2}=\left[\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right]\sin\theta+\left(2\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}\right)\cos\theta$$

Prove also that $x\frac{dy}{dt}-y\frac{dx}{dt}=r^2\frac{d\theta}{dt}$.

17. If $z=xyf\left(\frac{y}{x}\right)$, show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$.

Show that if z is constant, then

$$\frac{f'\left(\frac{y}{x}\right)}{f\left(\frac{y}{x}\right)}=\frac{x\left(y+x\frac{dy}{dx}\right)}{y\left(y-x\frac{dy}{dx}\right)}.$$

*18. If $f(x, y)=\dfrac{xy}{\sqrt{x^2+y^2}}$ for $x\neq 0$, $y\neq 0$ and $f(0, 0)=0$, show that $f(x, y)$ does not possess a first differential at the origin.

*19. Explain why the theorem of the total differential does not hold for the function

$z=x \sin\left(4 \tan^{-1}\frac{y}{x}\right)$ where $z=0$ for $x=0$, $y=0$, at the origin.

12. Application to small errors. It has already been shown that if $y=f(u, v)$ where u and v are independent variables,

$$\delta y=\frac{\partial f}{\partial u}\delta u+\frac{\partial f}{\partial v}\delta v+\epsilon_1\,\delta u+\epsilon_2\,\delta v,$$

where ϵ_1 and ϵ_2 approach zero with δu and δv. Consequently the above equation can be *approximately* written as

$$\delta y=\frac{\partial f}{\partial u}\delta u+\frac{\partial f}{\partial v}\delta v;$$

so that if δu and δv are small errors in u and v the error in δy is approximately

$$\frac{\partial f}{\partial u}\delta u+\frac{\partial f}{\partial v}\delta v.$$

Ex. 1. A quantity y is derived from measurement of a quantity x by means of a relation $y=f(x, t)$ where t is a parameter connected with x by the relation $\phi(x, t)=0$. Show that if an error δx is made in the estimation of x, then the resulting error δy in the computed value of y is given approximately by the formula

$$\delta y=\frac{\frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial t}-\frac{\partial f}{\partial t}\cdot\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial t}}\delta x.$$

Here $\delta y=\frac{\partial f}{\partial x}\delta x+\frac{\partial f}{\partial t}\delta t$

and $\frac{\partial \phi}{\partial x}\delta x+\frac{\partial \phi}{\partial t}\delta t=0$, approximately.

Therefore $\delta y=\frac{\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial t}-\frac{\partial f}{\partial t}\frac{\partial \phi}{\partial x}}{\frac{\partial \phi}{\partial t}}\delta x$, approximately.

Ex. 2. The luminosity L of a star is connected with its mass M by the relation $L=a\ M(1-\beta)$, where β is a positive number less than unity connected with M by the relation $1-\beta=b\beta^{4}M^{2}$, a and b being given constants.

If p is the percentage error made in the estimate of M, express the resulting percentage error in the calculated luminosity in terms of p and β and show that it lies between p and $3p$.

Here $\delta L=a(1-\beta)\ \delta M-a\ M\delta\beta$

and $-\delta\beta=4M^{2}b\beta^{3}\delta\beta+2b\beta^{4}M\delta M$

or $-\delta\beta(1+4b\beta^{3}M^{2})=2b\beta^{4}M\delta M.$

Therefore $\delta L=a(1-\beta)\delta M+aM\frac{2b\beta^{4}M\delta M}{1+4b\beta^{3}M^{2}}$

$$=\frac{(1-\beta)(1+4b\beta^{3}M^{2})+2M^{2}b\beta^{4}}{1+4b\beta^{3}M^{2}}\delta Ma$$

$$=\frac{(1-\beta)(1+4b\beta^{3}M^{2})+2(1-\beta)aMp}{(1+4b\beta^{3}M^{2})\quad 100}$$

Whence percentage error p_1 in L is given by

$$p_1=\frac{(1-\beta)(3+4b\beta^{3}M^{2})}{(1+4b\beta^{3}M^{2})}\cdot\frac{ap}{L}$$

$$=\frac{(1-\beta)(3+4\beta^{3}M^{2})}{(1+4b\beta^{3}M^{2})}\quad\frac{apM}{aM(1-\beta)}.$$

$$=p+\frac{2p}{1+4b\beta^3 M},$$

which clearly lies between p and $3p$.

Ex. 3. A triangle ABC is determined from observed values of b, c and A, but it is afterwards found that there are small errors δb, δc and δA in these quantities. Prove that the consequent error in R, the radius of the circle ABC is given approximately by

$$\frac{\delta R}{R}=\frac{\cos C\,.\,\delta b+\cos B\,.\,\delta c}{a}+\frac{\cos B\cos C}{\sin A}\delta A.$$

We know that

$$a^2=b^2+c^2-2bc\cos A \text{ and } R=\frac{a}{2\sin A}.$$

$$\therefore\ R^2=\frac{a^2}{4\sin^2 A}=\frac{b^2+c^2-2bc\cos A}{4\sin^2 A}.$$

Taking logarithms of both sides

$$2\log R=\log(b^2+c^2-2bc\cos A)-\log 4-2\log\sin A.$$

If b, c, A change to $b+\delta b$, $c+\delta c$, A$+\delta$A, the consequent change in R is given approximately on differentiating this equation totally. We thus get

$$2\frac{\delta R}{R}=\frac{2b.\delta b+2c.\delta c-2c\cos A.\,\delta b-2b\cos .A\delta c+2bc\sin A.\delta A}{b^2+c^2-2bc\cos A}$$

$$-2\frac{\cos A}{\sin A}\delta A.$$

$$\therefore\ \frac{\delta R}{R}=\frac{(b-c\cos A)\,\delta b+(c-b\cos A)\delta c+bc\sin A\,.\,\delta A}{a^2}$$

$$-\frac{\cos A}{\sin A}.\delta A$$

$$=\frac{a\cos C\,.\,\delta b+a\cos B\,.\,\delta c}{a^2}+\left(\frac{bc\sin A}{a^2}-\frac{\cos A}{\sin A}\right).\delta A$$

$$=\frac{\cos C\,.\,\delta b+\cos B\,.\,\delta c}{a}+\left(\frac{\sin B\sin C}{\sin A}-\frac{\cos A}{\sin A}\right).\delta A$$

But $\sin B\sin C-\cos A=\sin B\sin C+\cos(B+C)$

$$=\cos B\cos C.$$

$$\therefore \quad \frac{\delta R}{R} = \frac{\cos C \,.\, \delta b + \cos B \,.\, \delta c}{a} + \frac{\cos B \cos C}{\sin A} \,.\, \delta A.$$

EXAMPLES XIV

1. The error in the area A of an ellipse due to small errors in the lengths of the semi-axes a, b is given by

$$\frac{\delta A}{A} = \frac{\delta a}{a} + \frac{\delta b}{b}.$$

2. Two sides a, b of a triangle and the included angle C are measured ; find the error in the computed length of the third side c due to small errors δa, δb, δC in the observed values of two sides and the included angle.

3. If the area of a triangle be calculated by measuring the side a and the angles B and C, show that the proportional error in the area, due to small errors in the measurement is given by

$$\frac{\delta \triangle}{\triangle} = 2\,\frac{\delta a}{a} + \frac{c}{a}\,\frac{\delta B}{\sin B} + \frac{b}{a}\,\frac{\delta C}{\sin C}.$$

4. The area of a triangle ABC is calculated by measuring a, b and C ; show that the relative error in area is given by

$$\frac{\delta \triangle}{\triangle} = \frac{\delta a}{a} + \frac{\delta b}{b} + \cot C\, \delta C.$$

5. A triangle ABC is determined from a, b, A ; prove that if A receive an increment δA, the area S receives an increment δS, where

$$\frac{\delta S}{S} = -\frac{\cos C}{\sin A \cos B}\,\delta A.$$

6. The side a and the opposite angle A of a triangle ABC remain constant ; show that when the other sides and angles are slightly varied,

$$\frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0.$$

7. If a triangle ABC is slightly altered but so as to remain inscribed in the same circle, prove that

$$\frac{\delta a}{\cos A} + \frac{\delta b}{\cos B} + \frac{\delta c}{\cos C} = 0.$$

8. In measuring two sides of a triangle which include an angle of 30°, one side is found to be 27 inches with a possible error 0·10 inch, and the other 13 inches with a possible error of 0.05 inch. What is an approximate value for the largest possible error in the area of the triangle due to the errors in measuring the sides ?

9. The sides of an acute-angled triangle are measured. Prove that the increment in A due to small increments in the sides a, b, c is given by the equation

$$bc \sin A\, \delta A = -a(\cos C\, \delta b + \cos B\, \delta c - \delta a).$$

Supposing that the limits of error in the length of any side are $\pm\mu$ per cent., where μ is small, prove that the limits of error in A are approximately

$$\pm 1{\cdot}15 \frac{\mu a^2}{bc \sin A} \text{ degrees.}$$

10. The area of a triangle whose sides are a, b, c is Δ. Prove that the error in the area corresponding to error δa, δb, δc in the sides is approximately given by

$$2\Delta\, \delta\Delta = s^2 \delta p - s\, \delta q - abc\, \delta s,$$

where

$$2s = a+b+c$$

$$2p = a^2+b^2+c^2$$

$$3q = a^3+b^3+c^3.$$

11. If the density ρ of a body be calculated from its weights W, w in air and in water respectively, show that the relative error in ρ due to errors δ W, δw in W and w is given by

$$\frac{\delta \rho}{\rho} = \frac{-w}{W-w} \frac{\delta W}{W} + \frac{\delta w}{W-w}.$$

12. In a tangent galvanometer, the tangent of the deflection of the needle is proportional to the current: show that the relative error in the value of current due to an error in the reading of the deflection is least when the deflection is 45°.

13. A, B are two points, a feet apart, on a horizontal plane in line with the base C of a tower. The elevations α, β of the top of the tower are observed at A and B ($\beta > \alpha$).

If x is the distance of C from the middle point of AB prove that the increment δx due to small increments $\delta \alpha$, $\delta \beta$ in α, β is given by

$$\frac{\delta x}{x} = \frac{\sin 2\beta\, \delta\alpha - \sin 2\alpha\, \delta\beta}{\sin(\beta+\alpha)\sin(\beta-\alpha)}.$$

13. Euler's Theorem on Homogeneous Functions.

A function $f(x, y)$ is said to be **homogeneous** and of order n if $f(tx, ty) = t^n f(x, y)$, whatever t may be. This definition can be easily extended to a function of any number of varibles.

If $u = f(x, y)$ be a homogeneous function of order n and possess continuous partial derivatives, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu.$$

Since $f(x, y)$ is homogeneous and of order n, therefore whatever t may be

$$f(tx, ty) = t^n f(x, y).$$

Let $tx = u$ and $ty = v$, so that

$$f(u, v) = t^n (x, y).$$

Differentiating both sides with respect to t, we get

$$\frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{dv}\frac{dv}{dt} = nt^{n-1} f(x, y)$$

or
$$x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} = n\, t^{n-1} f(x, y). \quad \ldots\ldots(1)$$

As this is true whatever t may be, let $t = 1$; so that this equation becomes
$$x\frac{\partial u}{\partial u} + y\frac{\partial u}{\partial y} = nu. \quad \ldots\ldots(2)$$

This theorem can be similarly proved for any number of variables and is called Euler's Theorem on Homogeneous Functions.

Differentiating (1) again with regard to t and then putting $t = 1$ we can show that

$$x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial xy} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1) f(x, y),$$

which can be symbolically put as

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^2 f = n(n-1)\, f(x,\ y).$$

More generally, it can be shown that

$$\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^m f = n(n-1)\ldots(n-m+1)\, f(x,\ y).$$

The converse of Euler's Theorem is also true, that is to say, if a function $f(x, y)$ satisfies the equation $x\frac{\partial f}{\partial x} + y\frac{\partial}{\partial y} = n f(x, y)$, then it must be homogeneous and of order n. For proof the reader is referred to Forsyth's Differential Equations.

Note. It may be remarked that by $\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n$ we do not here mean that the operator $\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)$ is repeated n times but it stands for $x^n \frac{\partial^n}{\partial x^n} + n\, x^{n-1} y \frac{\partial^n}{\partial x^{n-1}\partial y} + \cdots\cdots + y^n \frac{\partial^n}{\partial y^n}$.

Ex. 1. Verify Euler's Theorem for the function

$$u = x^4 \log \frac{x^{\frac{1}{4}} - y^{\frac{1}{4}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}}$$

$$u = x^4 \log \frac{x^{\frac{1}{4}} - y^{\frac{1}{4}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}} = x^4 \log (x^{\frac{1}{4}} - y^{\frac{1}{4}}) - x^4 \log (x^{\frac{1}{4}} + y^{\frac{1}{4}})$$

$$\frac{\partial u}{\partial x} = 4x^3 \log (x^{\frac{1}{4}} - y^{\frac{1}{4}}) + x^4 \frac{\frac{1}{4}x^{-\frac{3}{4}}}{x^{\frac{1}{4}} - y^{\frac{1}{4}}} - 4x^3 \log (x^{\frac{1}{4}} + y^{\frac{1}{4}}) - \frac{x^4 \cdot \frac{1}{4}x^{-\frac{3}{4}}}{x^{\frac{1}{4}} + y^{\frac{1}{4}}}.$$

$$\frac{\partial u}{\partial y}=x^4\frac{-\frac{1}{4}y^{-\frac{3}{4}}}{x^{\frac{1}{4}}-y^{\frac{1}{4}}}-x^4\frac{\frac{1}{4}y^{-\frac{3}{4}}}{x^{\frac{1}{4}}-y^{\frac{1}{4}}}.$$

$$\therefore\ x\frac{\partial u}{\partial x}=4x^4\log\frac{x^{\frac{1}{4}}-y^{\frac{1}{4}}}{x^{\frac{1}{4}}+y^{\frac{1}{4}}}+\frac{1}{4}x^{\frac{1}{4}}\left(\frac{1}{x^{\frac{1}{4}}-y^{\frac{1}{4}}}-\frac{1}{x^{\frac{1}{4}}+y^{\frac{1}{4}}}\right)$$

$$=4x^4\log\frac{x^{\frac{1}{4}}-y^{\frac{1}{4}}}{x^{\frac{1}{4}}+y^{\frac{1}{4}}}+\frac{1}{2}\frac{x^{\frac{1}{4}}y^{\frac{1}{4}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}} \qquad \ldots\ldots(1)$$

$$\text{and } y\frac{\partial u}{\partial y}=-\frac{1}{4}x^4y^{\frac{1}{4}}\left(\frac{1}{x^{\frac{1}{4}}-y^{\frac{1}{4}}}+\frac{1}{x^{\frac{1}{4}}+y^{\frac{1}{4}}}\right)$$

$$=-\frac{1}{2}\frac{x^{\frac{1}{4}}y^{\frac{1}{4}}}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}. \qquad \ldots\ldots(2)$$

Adding (1) and (2) we get

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=4x^4\log\frac{x^{\frac{1}{4}}-y^{\frac{1}{4}}}{x^{\frac{1}{4}}+y^{\frac{1}{4}}}=4u.$$

Ex. 2. If $u=\tan^{-1}\frac{x^3+y^3}{x-y}$, verify that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\sin 2u.$$

We have $\tan u=\frac{x^3+y^3}{x-y}$, which shows that $\tan u$ is a homogeneous function of the second degree. Applying Euler's Theorem to tan, we get

$$x\frac{\partial}{\partial x}(\tan u)+y\frac{\partial}{\partial y}(\tan u)=2\tan u.$$

$$\text{i.e.,}\qquad x\frac{\partial u}{\partial x}\sec^2u+y\frac{\partial u}{\partial y}\sec^2u=2\tan u.$$

$$\therefore\qquad x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{2\tan u}{\sec^2 u}=\sin 2u.$$

EXAMPLES XV

1. If $v=(ax^3-by^3)^{\frac{1}{3}}$, verify that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=u$.

2. If $z=f\left(\frac{y}{x}\right)$, prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=0$.

3. If $z=\frac{xy}{x+y}$, verify that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$

and $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\,\partial y}+y^2\frac{\partial^2 z}{\partial y}=0$.

4. If $z=(ax^2+by^2)^{\frac{1}{2}}$ verify that

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0.$$

5. If $u=\frac{x+y}{x^2+y^2}$, verify that u is of order -1.

6. If $u=\tan^{-1}\frac{y}{x}$, verify by the converse of Euler's Theorem that the order of u is 0; also verify that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0,$$

7. Verify that the order of $u=\frac{x^{\frac{1}{2}}+y^{\frac{1}{2}}}{x^n+y^n}$ is $\frac{1}{2}-n$.

8. If $z=x^2\tan^{-1}\left(\frac{y}{x}\right)-y^2\tan^{-1}\left(\frac{x}{y}\right.$ prove that

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{x^2-y^2}{x^2+y^2},$$

9. If $u=(1-2xy+y^2)^{-\frac{1}{2}}$ prove that

$$\frac{\partial}{\partial x}\left[\ 1-x^2)\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left(y^2\frac{\partial u}{\partial y}\right)=0.$$

10. If $u=\sin^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, prove that

$$x\frac{\partial u}{\partial x}+\frac{\partial u}{y\partial y}=\tfrac{1}{2}\tan u.$$

11. If $u=f(r)$, where $r=\sqrt{x^2+y^2}$, show that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

12. If $u=f(x^2+y^2+z^2)$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=4(x^2+y^2+z^2)f''(x^2+y^2+z^2)+6f'(x^2+y^2+z^2).$$

13. If $u=x^n f\left(\frac{y}{x}\right)$ verify that

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\,\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u.$$

14. Given the equations : $xu^2+yv^3=xy^2$, $yu^3,-xv^2=x^2y$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$.

[Differentiating these equations with respect to x and y :

$$u^2+2ux\frac{\partial u}{\partial x}+3yv^2\frac{\partial v}{\partial x}-y^2=0 \quad \ldots(i)$$

$$2xu\frac{\partial u}{\partial y}+v^3+3v^2y\frac{\partial v}{\partial y}-2xy=0 \quad \ldots(ii)$$

$$3u^2y\frac{\partial u}{\partial y}-v^2-2xv\frac{\partial v}{\partial x}-2xy=0 \quad \ldots(iii)$$

$$u^3+3u^2y\frac{\partial u}{\partial y}-2xv\frac{\partial v}{\partial y}-x^2=0 \quad \ldots(iv)$$

(i) and (iii) give $\frac{\partial u}{\partial x}$, and $\frac{\partial v}{\partial x}$, and ($ii$) and ($iv$) give $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial y}$].

15. Given the equations : $x^3+xv=y$, $v^3+yu=x$, find $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$, $\frac{\partial v}{\partial y}$.

16. $u=f(H_n)$ where H_n is a homogeneous function of nth degree ; also suppose that we get from this equation $H_n=F(u)$. Then show that $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{F(u)}{F'(u)}$.

CHAPTER V

TAYLOR'S THEOREM

1. *Let $f(x)$ and its derivatives up to a certain order $n-1$ be continuous functions of x for all values of x lying between a and $a+h$ and let the nth derivative exist.* The formula of Taylor has for its object *to expand $f(a+h)$* in *ascending powers of h up to order n and the expansion has the form*

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)$$

$$+\ldots\ldots+\frac{h^{n-1}}{\lfloor n-1}f^{n-1}(a)+\frac{h^n}{\lfloor n}\mathrm{M}, \qquad \ldots(i)$$

where M is defined by this equation and is a function of h supposed different from zero.

2. Lagrange's Form of Remainder. The last term $\frac{h^n}{\lfloor n}\mathrm{M}$ of the formula (i) is called the remainder after n terms and is usually denoted by $\mathrm{R_n}$. *We shall now find some expression for R_n*

Consider the function

$$\phi(x)\equiv f(a+h)-f(x)-(a+h-x)f'(x)-\frac{(a+h-x)^2}{\lfloor 2}f''(x)-\ldots\ldots$$

$$-\frac{(a+h-x)^{n-1}}{\lfloor n-1}f^{n-1}(x)-\frac{(a+h-x)^n}{\lfloor n}\mathrm{M},$$

where M is defined by (i).

Now $\phi(a)=f(a+h)-f(a)-hf'(a)-\ldots\ldots$

$$-\frac{h^{n-1}}{\lfloor n-1}f^{n-1}(a)-\frac{h^n}{\lfloor n}\mathrm{M}=0,$$

by virtue of (i) and $\phi(a+h)=0$ by actual substitution.

Hence by Rolle's Theorem $\phi'(a+\theta h)=0$, $0<\theta<1$.

But $\phi'(x)=-\frac{(a+h-x)^{n-1}}{\lfloor n-1}\left(\mathrm{M}-f^n(x)\right)$

therefore $\phi'(a+\theta h)=0$, gives

$$\frac{(h-\theta h)^{n-1}}{\underline{n-1}}\left(\mathrm{M}-f^n(a+\theta h)\right)=0, \text{ or } \qquad \mathrm{M}=f^n(a+\theta h).$$

Therefore (*i*) takes the form

$$f(a+h)=f(a)+f'(a)+\frac{h^2}{\underline{2}}f''(a)$$

$$+\ldots+\frac{h^{n-1}}{\underline{n-1}}f^{n-1}(a)+\frac{h^n}{\underline{n}}f^n(a+\theta h). \quad \ldots(1)$$

This is called Taylor's Theorem with **Lagrange's Form of the Remainder.**

3. **Cauchy's Form of Remainder.** On the other hand if the expansion of $f(a+h)$ be taken in the form

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\underline{2}}f''(a)$$

$$+\ +\frac{h^{n-1}}{\underline{n-1}}f^{n-1}(a)+h\,\mathrm{M}, \qquad \ldots\ldots(ii)$$

where M is defined by this equation, then M can be found in a different form.

Consider the function

$$\phi(x)\equiv f(a+h)-f(x)-(a+h-x)f'(x)-\frac{(a+h-x)^2}{\underline{2}}f''(x)$$

$$-\ldots\ldots\ldots-\frac{(a+h-x)^{n-1}}{\underline{n-1}}f^{n-1}(x)-(a+h-x)\,\mathrm{M}.$$

Here $\phi(a)=0$ because of (*ii*), and $\phi(a+h)=0$, by actual substitution therefore by Rolle's Theorem $\phi'(a+\theta h)=0$, $0<\theta<1$.

But $\phi'(x)=-\frac{(a+h-x)^{n-1}}{\underline{n-1}}f^n(x)+\mathrm{M}$;

therefore $\phi'(a+\theta h)=0$, gives

$$M=\frac{h^{n-1}\ (1-\theta)^{n-1}}{\lfloor n-1}\ f^n\ (a+\theta h).$$

Substituting this value of M in (*ii*), we get

$$f(a+h)=f(a)+h\,f'(a)+\frac{h^2}{\lfloor 2}\,f''(a)$$

$$+\ldots\ldots+\frac{h^{n-1}}{\lfloor n-1}\,f^{n-1}(a)+\frac{h^n}{\lfloor n-1}\ (1-\theta)^{n-1}\,f^n(a+\theta h) \qquad \ldots\ldots(2)$$

This is called Taylor's Theorem with **Cauchy's Form of the Remainder.**

4. It has already been shown that if $\phi(x)$ is continuous for all values of x lying between a and $a+h$, and if $\phi'(x)$ exists, then

$$\phi(a+h)=\phi(a)+h\phi'\ (a+\theta h),\ 0<\theta<1 \qquad \ldots\ldots(3)$$

(Mean Value Theorem)

Let $\phi(x)=f(x)+(a+h-x)f'(x)+\frac{(a+h-x)^2}{\lfloor 2}\,f''(x)$

$$+\ldots\ldots+\frac{(a+h-x)^{n-1}}{\lfloor n-1}\,f^{n-1}x\Big)$$

Then $\phi(a+h)=f(a+h)$

and $\phi(a)=f(a)+h\,f'(a)+\frac{h^2}{\lfloor 2}f''(a)+\ldots\ldots+\frac{h^{n-1}}{\lfloor n-1}\,f^{n-1}\,(a).$

Also $\phi'(x)=\frac{(a+h-x)^{n-1}}{\lfloor n-1}\,f^n\,(x)$, other terms cancelling.

$$\therefore\ \phi'\ (a+\theta h)=\frac{(h-\theta h)^{n-1}}{\lfloor n-1}\,f^n(a+\theta h).$$

Therefore (3) takes the form

$$f(a+h)=f(a)+h\,f'(a)+\frac{h^2}{\lfloor 2}\,f''(a)+\ldots$$

$$+\frac{h^{n-1}}{\lfloor n-1}\,f^{n-1}\,(a)+\frac{h^n}{\lfloor n-1}\ (1-\theta)^{n-1}\,f^n\,(a+\theta h) \qquad \ldots\ldots(4)$$

As before, the last term

$R_n = \frac{h^n(1-\theta)^{n-1}}{\lfloor n-1} f^n(a+\theta h)$ is called Cauchy's Form of the Remainder.

When $f(x)$ is indefinitely derivable, we can take n as large as we like and send back R_n as far as we like and if R_n tends to zero when n tends to infinity, we can prolong the formula indefinitely, the last term disappearing in the limit and we get the expansion of $f(a+h)$ as an infinite series, namely

$$f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a)+\ldots\ldots+\frac{h^n}{\lfloor n}\,^n(a)+\ldots\ldots$$

5. Maclaurin's Formula. If in (1) we take $a=0$, $h=x$, we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\lfloor 2}f''(0)$$
$$+\ldots\ldots+\frac{x^{n-1}}{\lfloor n-1}f^{n-1}(0)+\frac{x^n}{\lfloor n}f^n(\theta x)$$

Similarly from (4) we get

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\lfloor 2}\,^n(0)+\ldots\ldots$$
$$+\frac{x^{n-1}}{\lfloor n-1}f^{n-1}(0)+\frac{x^n}{\lfloor n}(1-\theta)^{n-1}f^n(\theta x).$$

When the derivatives of $f(x)$ are determinate and continuous up to any order whatsoever and when the last term tends to zero as n tends to infinity, Maclaurin's Formula may be written as

$$f(x)=f(0)+xf'(0)+\frac{x^2}{\lfloor 2}f''(0)+\ldots+\frac{x^n}{\lfloor n}f^n(0)+\ldots\ldots$$

Applications. I. e^x. Let $f(x)=e^x$ be required to be expanded in powers of x. $\quad ^n(x)=e^x$ and $f^n(0)=1$.

Therefore Maclaurin's Formula with Lagrange's form of the Remainder gives

$$e^x=1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\ldots+\frac{x^{n-1}}{\lfloor n-1}+R_n,$$

where $R_n=\frac{x^n}{\lfloor n}\,e^{\theta x}$.

Now whatever x may be, $\frac{x^n}{\lfloor n}$ tends to zero when n tends to infinity and consequently R_n also tends to zero. Hence we get

$$\epsilon^x = 1 + x + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \ldots\ldots$$

II. a^x. If we change x into $x \log a$ in the preceding formula, we get

$$a^x = 1 + x \log a + \frac{(x \log a)^2}{\lfloor 2} + \ldots\ldots$$

III. sin x. If $f(x) = \sin x$, we have $f^n(x) = \sin\left(x + n\frac{\pi}{2}\right)$

For $x=0$, the values of $f(x)$ and its successive derivatives form a periodic suite of four terms 0, 1, 0, -1 ; 0, 1, 0, -1 ; .....If we take $n = 2m+1$,

$$f^{2m+1}\theta x = \sin\left[\theta x + (2m+1)\frac{\pi}{2}\right] = (-1)^m \cos\theta x.$$

By Maclaurin's Theorem with Lagrange's Form of the Remainder, we get

$$\sin x = \frac{x}{1} - \frac{x^3}{\lfloor 3} + \frac{x^5}{\lfloor 5} - \ldots\ldots + (-1)^{m-1}\frac{x^{2m-1}}{\lfloor 2m-1} + (-1)^m \frac{x^{2m+1}}{\lfloor 2m+1} \cos\theta x.$$

As in the case of e^x, the remainder approaches zero as n approaches infinity and therefore sin x can be expanded in powers of x to infinity.

$$\sin x = x - \frac{x^3}{3} + \frac{x^5}{\lfloor 5} - \ldots\ldots$$

IV. cos x. Similarly it can be shown that cos x can be expanded in powers of x :

$$\cos x = 1 - \frac{x^2}{\lfloor 2} + \frac{x^4}{\lfloor 4} - \frac{x^6}{\lfloor 6} + \ldots = \sum_0^\infty (-1)^m \frac{x^{2m}}{\lfloor 2m}$$

V. log (1+x). Taking $f(x)=\log(1+x)$, we get

$$f'(x)=\frac{1}{1+x} \quad \text{and} \quad f^n(x)=(-1)^{n-1}\frac{n-1}{(1+x)^n}.$$

$$\therefore \quad \log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\ldots\ldots+(-1)^{n-2}\frac{x^{n-1}}{n-1}+R_n,$$

where by Lagrange's Formula $R_n=(-1)^{n-1}\frac{x^n}{n}\left(\frac{1}{1+\theta x}\right)^n$.

and by Cauchy's Formula $R_n=(-1)^{n-1}\frac{x^n(1-\theta)^{n-1}}{(1+\theta x)^n}$.

In these formulae it is assumed that $x>-1$, for otherwise the function and the derivatives shall not be determinate in the interval $(0, x)$ and the formula of Maclaurin will not be applicable. If x is positive and ≤ 1, the formula of Lagrange shows that $|R_n|$ is less than $\frac{1}{n}$. If x is negative but greater than a fixed number $-r$($-r$ being >-1), the formula of Cauchy shows that $|R_n|<\frac{r^n}{1-r}$.

Therefore in both the cases, the remainder R_n approaches zero as n approaches infinity. Hence $\log(1+x)$ can be expanded in powers of x to infinity and we get

$$\log(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\ldots$$

If we put $x=1$, we have

$$\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots\ldots$$

VI. $(1+x)^m$. Binomial Theorem.

Taking $f(x)=(1-x)^m$, we get

$$f^n(x)=m(m-1)(m-2)\ldots\ldots(m-n+1)(1+x)^{m-n}.$$

We have therefore

$$(1+x)^m=1+mx+\frac{m(m-1)}{|2}x^2+\ldots\ldots+\frac{m(m-1)\ldots(m-n+2)}{|n-1}x^{n-1}+R_n.$$

By Cauchy's Formula

$$R_n=\frac{m(m-1)(m-2)\ldots(m-n+1)}{\underline{|n-1}}x^n(1-\theta)^{n-1}(1+\theta x)^{m-n}$$

$$=(1+\theta x)^{m-1}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\frac{m(m-1)\ldots(m-n+1)}{\underline{|n-1}}x^n$$

$$=(1+\theta x)^{m-1}\left(\frac{1+\theta}{1+\theta x}\right)^{n-1}U_n$$

where U_n stands for $\dfrac{m(m-1)\ldots(m-n+1)}{\underline{|n-1}}x^n$

It is needless to remark that if m is a positive integer the expansion stops at the stage when $n=m+1$, because the factor $m-n+1$ is zero. If m is fractional or negative, then the expansion does not stop. If x is negative it must be >-1, for otherwise the derivative $f^n(x)$ will not be continuous in the interval $(0, x)$ when $n>m$.

Now when $|x|<1$, when it is clear that $0<1-\theta<1+\theta x$ and consequently $\dfrac{1-\theta}{1+\theta x}$ is a proper fraction and $\left(\dfrac{1-\theta}{1+\theta x}\right)^{n-1}$ must remain less than unity, however large n may be.

Therefore $|R_n|<|(1+\theta x)^{m-1}|\times|U_n|$.

But it is known that when $|x|<1$, U_n tends to zero when n tends to infinity. Hence R_n tends to zero when n tends to infinity. Hence the expansion to infinity is justified when $|x|<1$.

6. Calculation of the nth derivative. The practical difficulty in the application of Maclaurin's Theorem lies in the calculation of $f^n(x)$; in fact there are only a few cases in which the nth derivative can be expressed in a controllable form ; and consequently the discussion of the remainder R_n offers great difficulties. In particular cases, however, we can find $f^n(0)$ and the expansion then can be put down.

In this connection Leibnitz's Theorem is very useful.

As an illustration let us take,

Ex. 1. $y=f(x)=\sin(m\sin^{-1}x)$ (i)

$$y_1=f'(x)=\frac{m}{\sqrt{1-x^2}}\cos(m\sin^{-1}x) \quad (ii)$$

$$\therefore\ y_1^2=\frac{m^2}{(1-x^2)}(1-y^2)$$

or $(1-x^2)y_1^2=m^2(1-y^2)$.

Differentiating again, we get

$$2y_1y_2(1-x^2)+y_1^2(-2x)=-2m^2yy_1$$

or $(1-x^2)f''(x)-xf'(x)=-m^2f(x)$. *(iii)*

By putting $x=0$ in (i), (ii) and (iii), we get

$f(0)=0,\ f'(0)=m,\ f''(0)=0$. *(iv)*

Now differentiating (iii) n times by Leibnitz's Theorem we have

$$f^{n+2}(x)(1-x^2)+nf^{n+1}(x)(-2x)+\frac{n(n-1)}{\lfloor 2}f^n(x)(-2)$$

$$-f^{n+1}(x)\,x-nf^n(x)=-m^2f^n(x)$$

Or $(1-x^2)f^{n+2}(x)-(2n+1)xf^{n+1}(x)-(n^2-m^2)f^n(x)=0$

and therefore when $x=0$

$f^{n+2}(0)=(n^2-m^2)f^n(0)$.

From (v) by taking $n=1, 2, 3\ldots\ldots$ we find in succession all the derivatives above the second for $x=0$, because we know the first two.

$f^4(0)=(2^2-m^2)f''(0)=0$

$f^6(0)=(4^2-m^2)f^4(0)=0,$

and so on ; *thus all even derivatives are zero.*

Again $f^3(0)=(1^2-m^2)f(0)=m(1^2-m^2)$

$f^5(0)=(3^2-m^2)f^3(0)=(3^2-m^2)(1^2-m^2)m$

and so on, the general value being

$f^{2n-1}(0)=m(1^2-m^2)(3^2-m^2)\ldots\ldots\{(2n-3)^2-m^2\}.$

Hence $\sin(m\sin^{-1}x)=mx+\frac{m(1^2-m^2)}{\lfloor 3}x^3$

$$+\frac{m(1^2-m^2)(3^2-m^2)}{\lfloor 5}x^5+\ldots\ldots$$

It is clear that if m is an odd integer, then the expansion ends after a certain stage, otherwise it extends to infinity.

Cor. 1. If we set $\sin^{-1}x=\theta$, so that $x=\sin\theta$, we get

$$\sin m\theta=m\sin\theta+\frac{m(1^2-m^2)}{\lfloor 3}\sin^3\theta$$

$$+\frac{m(1^2-m^2)(3^2-m^2)}{\lfloor 5}\sin^5\theta+\ldots\ldots$$

Cor. 2. Similarly we can get

$$\cos m\theta = 1 - \frac{m^2 \sin^2\theta}{\lfloor 2} + \frac{m^2(m^2-2^2)}{\lfloor 4} \sin^4\theta + \ldots\ldots$$

As a second illustration, let us take

Ex. 2. $y = f(x) = e^{a \sin^{-1} x}$ (*i*)

$$y_1 = f'(x) = e^{a \sin^{-1} x} \frac{a}{\sqrt{1-x^2}} = \frac{ay}{\sqrt{1-x^2}} \quad (ii)$$

$\therefore\ y_1^2(1-x^2) = a^2y^2.$

Differentiating, we get $2y_1y_2(1-x^2) - 2xy_1^2 = 2ayy_1$

or $y_2(1-x^2) - xy_1 = a^2y$

i.e., $f''(x)(1-x^2) - xf'(x) = a^2f(x).$ (*iii*)

Setting $x=0$ in (*i*), (*ii*) and (*iii*), we get

$f(0) = e^0 = 1,\ f'(0) = a,\ f''(0) = a^2.$

Differentiating (*iii*) n times by Leibnitz's Theorem, we get

$(1-x^2)f^{n+2}(x) - (2n+1)xf^{n+1}(x) - (n^2+a^2)f^n(x) = 0.$

Putting $x=0$, we have $f^{n+2}(0) = (n^2+a^2)f^n(0).$

Therefore putting $n = 1, 2, 3\ldots\ldots$ we get

$f^3(0) = (1^2+a^2)a$

$f^4(0) = (2^2+a^2)f''(0) = (2^2+a^2)a^2$

$f^5(0) = (3^2+a^2)f^3(0) = a(1^2+a^2)(3^2+a^2)$, and so on.

Hence $e^{a \sin^{-1} x} = 1 + ax + \frac{a^2}{\lfloor 2}x^2 + \frac{a(1^2+a^2)}{\lfloor 3}x^3$

$$+ \frac{a^2(2^2+a^2)}{\lfloor 4}x^4 + \ldots\ldots$$

Remarks. It may be noticed that $e^{a \sin^{-1} x}$ can also be expanded as

$$e^{a \sin^{-1} x} = 1 + (a \sin^{-1} x) + \frac{(a \sin^{-1} x)^2}{\lfloor 2} + \frac{(a \sin^{-1} x)^3}{\lfloor 3} + \ldots.$$

Comparing the powers of a, a^2 in both the expansions we get

$$\sin^{-1}x = x + \frac{1^2 x^3}{\lfloor 3} + \frac{1^2.3^2}{\lfloor 5}x^5 + \ldots\ldots$$

$$\frac{(\sin^{-1} x)^2}{\lfloor 2} = \frac{x^2}{\lfloor 2} + \frac{2^2}{\lfloor 4}x^4 + \frac{2^2.4^2}{\lfloor 6}x^6 + \ldots\ldots$$

and so on

EXAMPLES XVI

Assuming the possibility of expansion, prove the following results .—

1. $\tan^{-1}(x+h)=\tan^{-1}x+\frac{h}{1+x^2}-\frac{xh^2}{(1+x^2)^2}-\ldots\ldots$

2. $\sin^{-1}(x+h)=$

$$\sin^{-1}x+\frac{h}{\sqrt{1-x^2}}+\frac{x}{(1-x^2)^{3}}\frac{h^2}{\lfloor 2}+\frac{1+2x^2}{(1-x^2)^{\frac{5}{2}}}\frac{h^3}{\lfloor 3}+\ \ldots$$

3. $\log\sin(x+h)=\log\sin x+h\cot x$

$$+\frac{h^2}{\lfloor 2}\operatorname{cosec}^2 x+\frac{h^3}{\lfloor 3}\frac{\cos x}{\sin^3 x}+\ldots\ldots$$

4. $e^{a(x+h)}\sin m(x+h)=e^{ax}[\sin mx+h(a^2+m^2)^{\frac{1}{2}}\times$

$\sin(mx+\phi)+\frac{h^2}{\lfloor 2}(a^2+m)^{\frac{3}{2}}\sin(mx+3\phi)+\ldots\ldots]$

where $\tan\phi=\frac{m}{a}$.

5. $\tan^{-1}x=x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots\ldots$

6. $\tan x=x+\frac{x^3}{1.3}+\frac{2x^5}{1.3.5}+\ldots\ldots$

7. $e^{\sin x}=1+x+\frac{x^2}{1.2}-\frac{3x^4}{1.2.3.4}-\ldots\ldots$

8. $\frac{e^x}{\cos x}=1+x+\frac{2x^2}{\lfloor 2}+\frac{4x^3}{\lfloor 3}+\ldots\ldots$

9. $e^{ax}\cos bx=1+ax+\frac{a^2-b^2}{\lfloor 2}x^2+\frac{a(a^2-3b^2)}{\lfloor 3}x^3+\ldots\ldots$

$$\frac{(a^2-b^2)^{\frac{n}{2}}}{\lfloor n}x^n\cos\left(n\tan^{-1}\frac{b}{a}\right)+\ldots\ldots$$

showing that the expansion is always valid.

10. $e^{ax}\sin bx = bx + abx^2 + \frac{3a^2b - b^3}{6}x^3$

$$+ \ldots\ldots + \frac{(a^2+b^2)^{\frac{n}{2}}}{\lfloor n}x^n \sin\left(n\tan^{-1}\frac{b}{a}\right) + \ldots\ldots$$

11. Expand $\sin^{-1}\frac{2x}{1+x^2}$ in powers of x.

12. Prove that $f(mx) = f(x) + (m-1)\,xf'(x)$

$$+ \frac{(m-1)^2x^2}{\lfloor 2}f''(x) + \frac{(m-1)^3\,x^3}{\lfloor 3}f'''(x) + \ldots\ldots$$

13. Expand $\sin(e^{\theta}-1)$ up to and including the term in θ^4.

14. Show by Maclaurin's Theorem that the first five terms in the power series for $\log(1+\sin x)$ are

$$x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \frac{x^5}{24}.$$

15. Show that $\log\tan\left(\frac{\pi}{4}+x\right) = 2x + \frac{4}{3}x^3 + \frac{4}{5}x^5 + \ldots$

16. Show that $\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \ldots\ldots$

17. Expand $e^x\cos x$ and $e^x\sin x$ in powers of x, find the nth term and show that the expansions hold for all values of x.

18. If $y = \frac{1}{2}(\sin^{-1}x)^2 = a_0 + a_1x + a_2\frac{x^2}{\lfloor 2} + \frac{a_3x^3}{\lfloor 3} + \ldots\ldots$

show that $a_{n+2} - n^2a_n = 0$. Hence expand y.

19. If $y = \tan^{-1}x$, show that $y_1(1+x^2) = 1$; use the result to find an expansion for $\tan^{-1}x$.

20. If $y = \sin^{-1}x = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots\ldots$

prove that

(*i*) $(1-x^2)y_2 = xy_1$ (*ii*) $(n+1)(n+2)a_{n+2} = n^2a_n$

(*iii*) $\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1}{2}\cdot\frac{3}{4}\frac{x^5}{5} + \ldots\ldots$

21. Prove that $(x+\sqrt{(1+x^2)})^n = 1 + nx + \frac{n^2x^2}{\lfloor 2}$

$$+\frac{n(n^2-1^2)}{\underline{|3}}x^3+\frac{n^2(n^2-2^2)}{\underline{|4}}x^4+\frac{n(n^2-1^2)(n^2-3^2)}{\underline{|5}}x^5+\ldots\ldots$$

and deduce the expansions of

$$\log\ (x+\sqrt{1+x^2})\ \text{ and }\ \frac{1}{\underline{|2}}\ \{\log\ (x+\sqrt{x^2+1})\ \}^2.$$

22. Show that

$$\tfrac{1}{2}\{\log\ (1+x)\}^2=\frac{x^2}{2}-(1+\tfrac{1}{2})\frac{x^3}{3}\ (1+\tfrac{1}{2}+\tfrac{1}{3})\frac{x^4}{4}-\ldots$$

23. Show that if $e^x \sin x=a_0+a_1x+a_2\frac{x^2}{\underline{|2}}+\ldots+\frac{an}{\underline{|n}}x_n+$

then $a_{4n}=0$, $a_{4n+1}=(-1)^n4^n$.

24. Prove that

$$\log\ (1-x+x^2)=-x+\frac{x^2}{2}+\frac{2x^3}{3}+\frac{x^4}{4}-\frac{x^5}{5}-\frac{x^6}{3}\ldots\ldots$$

25. By means of the expansion of e^x and $\log\ (1+x)$ prove that

$$\left(1+\frac{1}{n}\right)^n=e\left(1-\frac{1}{2n}+\frac{11}{24n^2}-\frac{7}{16n^3}+\ldots\ldots\right)$$

when n is large.

26. Prove that $\dfrac{e^\theta}{\cos\ \theta}=1+\dfrac{\sin\ \theta}{\underline{|1}}+\dfrac{(1+1^2)\ \sin^2\ \theta}{\underline{|2}}$

$$+\frac{(1+2^2)\ \sin^3\ \theta}{\underline{|3}}+\frac{(1+1^2)(1+3^2)}{\underline{|4}}\ \sin^4\theta+\ldots$$

27. Given that $y^3-y+x=0$; by developing y in powers of x by Maclaurin's Theorem, show that y is equal to

$$x+x^3+3x^5+12x^7+\ldots$$

$$\text{or } 1-\frac{x}{2}-\frac{3x^2}{8}-\frac{8x^2}{16}-\ldots\ldots\text{or } -1-\frac{x}{2}+\frac{3x^2}{8}-\frac{8x^3}{16}+\ldots$$

28. If $y^3-6xy-8=0$, show that

$$y=2+x-\frac{1}{2}\frac{x^3}{\underline{|3}}+\frac{x^4}{\underline{|4}}+\ldots\ldots\ldots$$

29. Can $\sin 1/x$ or $\log x$ or $\cot x$ be expanded in ascending powers of x by Maclaurin's Theorem? Give reasons.

30. Given that $f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x+\theta h)$, and $f(x)=x^3$, find θ.

31. Given that $f(x)=f(0)+xf'(\theta x)$, $0<\theta<1$; deduce Maclaurin's Theorem.

32. Given that $\phi(a+h)=\phi(a)+h\phi(a+\theta h)$ where $0<\theta<1$, prove that the limiting value of θ, when h is diminished indefinitely is $\frac{1}{2}$.

$[f(a+h)=f(a)+hf'(a+\theta h) \qquad 0<\theta<1,$

also $f(a+h)=f(a)+hf'(a)+\frac{h^2}{\lfloor 2}f''(a+\theta_1.\theta h)$

$\therefore \quad f'(a+\theta h)=f'(a)+\frac{h}{2}f''(a+\theta_1,\theta h)$

$f'(a)+\theta h.f''(a+\theta_2.\theta h)=f'(a)+\frac{h}{2}f''(a+\theta_1.\theta h)$

$\therefore \qquad \theta=\frac{1}{2}$ when $h\to 0$. [See **Ex. 31**]

because $\underset{h\to 0}{\text{Lt}}\ f''(a+\theta_2\theta h)=\underset{h\to 0}{\text{Lt}}\ f''(a+\theta_1.\theta h)]$

33. In the equation $f(a+h)=f(a)+hf'(a)$
$+\frac{h^2}{\lfloor 2}f''(a)+\ldots..+\frac{h^{n-1}}{\lfloor n-1}f^{n-1}(a)+\frac{h^n}{\lfloor n}f^n(a+\theta h)$,
show that the limiting value of θ as h is indefinitely diminished is $1/(n+1$.

$\left[f(a+h)=f(a)+hf'(a)+\ldots..+\frac{h^n}{\lfloor n}f^n(a+\theta_n h)\right.$

also $f(a+h)+f(a)=hf'(a)+\ldots...$
$+\frac{h^n}{\lfloor n}f^n(a)+\frac{h^{n+1}}{\lfloor n+1}f^{n+}(a+\theta_{n+1}h)$,

where θ_n and θ_{n+1} lie between 0 and 1.

Equating

$f^n(a+\theta_n h)=f^n(a)+\frac{h}{n+1}f^{n+1}(a+\theta_{n+1}h)$,

Also $f^n(a+\theta_n h)=f^n(a)+\theta_n hf^{n+1}(a+\theta.\theta_n h)$, $0<\theta<1$;

Therefore $f^n(a)+\theta_n h f^{n+1}(a\theta.\theta_n h)=f^n(a)$

$$+\frac{h}{n+1}f^{n+1}(a+\theta_{n+1}h)$$

$$\therefore \qquad \theta_n f^{n+1}(a+\theta.\theta_n h)=\frac{f^{n+1}(a+\theta_{n+1}h)}{n+1}$$

$$\therefore \qquad \theta_n=\frac{\mathbf{1}}{\mathbf{n+1}} \text{ when } h\to 0$$

because $\underset{h\to 0}{\text{Lt}}\ f^{n+1}(a+\theta.\theta_n h)=\underset{h\to 0}{\text{Lt}}\ f^{n+1}(a+\theta_{n+1}h)$]

34. In the formula $f(x)=f(0)+xf'(0)+\frac{x^2}{\lfloor 2}f''(0)+\ldots.$

$$+\frac{x^{n-1}}{\lfloor n-1}f^{n-1}(0)+\frac{x^n}{\lfloor n}f^n(\theta x),$$

where θ is a positive proper fraction, show that when

$$f(x)=(1-x)^{-2} \text{ and } x=-1,$$

$$\theta=\left(\frac{4n+4}{2n+1}\right)^{\frac{1}{n+2}}-1.$$

35. Expand $(1-x)^{n+\frac{1}{2}}$ in the above form and prove that as x tends to unity the value of θ in $f^n(\theta x)$ tend to the limit

$$(4n+1)/(2n+1)^2.$$

CHAPTER VI

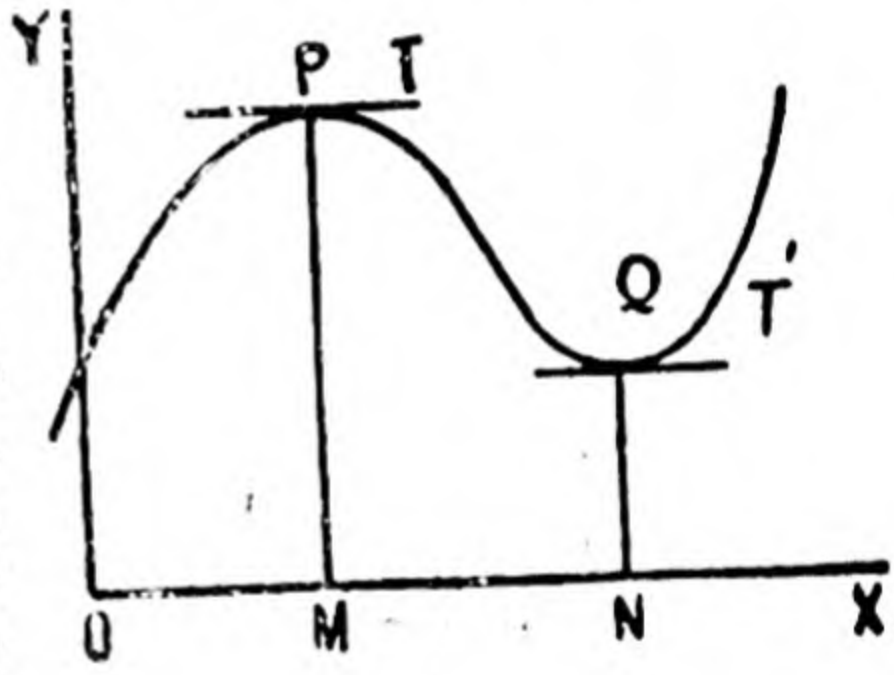
MAXIMA AND MINIMA

1. Definitions. Let $f(x)$ be a function of the variable x and c a particular value of x. If we can assign a positive quantity δ, however small, such that

$$f(c+h)<f(c)$$

for all the values of h lying between $-\delta$ and $+\delta$, then $f(c)$ is said to be a **maximum** of $f(x)$.

Similarly if $f(c+h) > f(c)$ for all values of h lying between $-\delta$ and $+\delta$, then $f(c)$ is said to be a **minimum** of $f(x)$. Maximum and minimum values are also called **extremum** or **turning** or **stationary values**. *Geometrically*, if we construct the curve $y=f(x)$ the maximum and minimum values of $f(x)$ correspond to points such as P and Q of the curve and the ordinate MP is the greatest and the ordinate NQ is the least of the ordinates in sufficiently small neighbourhoods of these ordinates.



It follows from the definitions that if, as x increases and passes through the value c, the function $f(x)$ increases, attains the value $f(c)$ at $x=c$ and then decreases, then $f(x)$ has a maximum value $f(c)$ at $x=c$; similarly if $f(x)$ decreases, attains the value $f(c)$ at $x=c$, and then increases, $f(x)$ has a minimum at $x=c$.

It is important to remark that by definition a maximum or a minimum value of $f(x)$ is *not necessarily the greatest or the least value of the function in every interval in which x varies, but solely a greatest or a least value in interval sufficiently small containing the value c.* Consequently a function may have several maximum and minimum values in any given interval

Therefore $f^n(a)+\theta_n h f^{n+1}(a\theta.\theta_n h)=f^n(a)$

$$+\frac{h}{n+1}f^{n+1}(a+\theta_{n+1}h)$$

$$\therefore \qquad \theta_n f^{n+1}(a+\theta.\theta_n h)=\frac{f^{n+1}(a+\theta_{n+1}h)}{n+1}$$

$$\therefore \qquad \theta_n=\frac{\mathbf{1}}{\mathbf{n+1}} \text{ when } h\to 0$$

because $\underset{h\to 0}{\text{Lt}}\ f^{n+1}(a+\theta.\theta_n h)=\underset{h\to 0}{\text{Lt}}\ f^{n+1}(a+\theta_{n+1}h)$]

34. In the formula $f(x)=f(0)+xf'(0)+\frac{x^2}{\lfloor 2}f''(0)+\ldots..$

$$+\frac{x^{n-1}}{\lfloor n-1}f^{n-1}(0)+\frac{x^n}{\lfloor n}f^n(\theta x),$$

where θ is a positive proper fraction, show that when

$$f(x)=(1-x)^{-2} \text{ and } x=-1,$$

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35. Expand $(1-x)^{n+\frac{1}{2}}$ in the above form and prove that as x tends to unity the value of θ in $f^n(\theta x)$ tend to the limit $(4n+1)/(2n+1)^2$.

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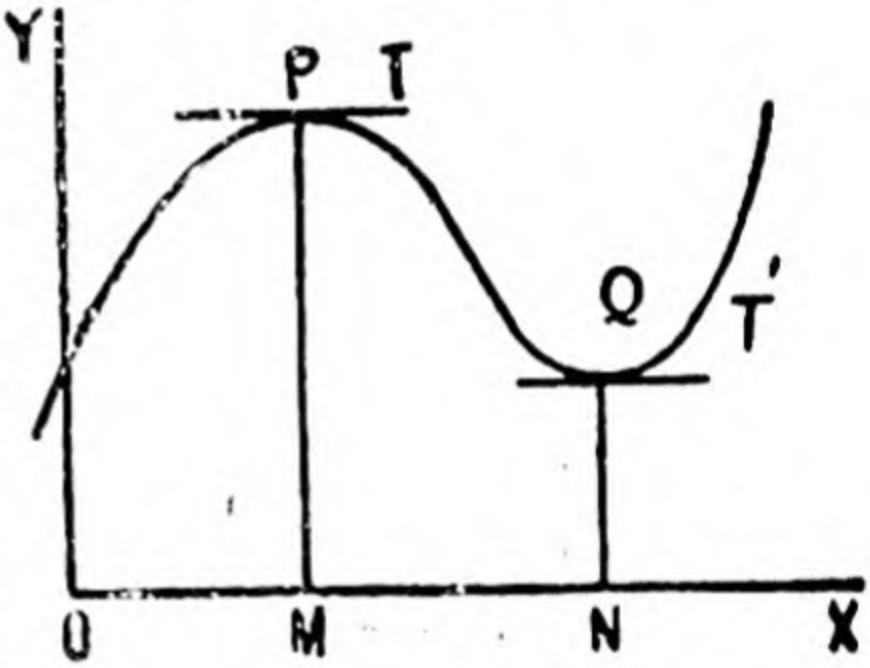


It follows from the definitions that if, as x increases and passes through the value c, the function $f(x)$ increases, attains the value $f(c)$ at $x=c$ and then decreases, then $f(x)$ has a maximum at $x=c$; similarly if $f(x)$ decreases, attains the value $f(c)$ at $x=c$, and then increases, $f(x)$ has a minimum at $x=c$.

It is important to remark that by definition a maximum or a minimum value of $f(x)$ is *not necessarily the greatest or the least value of the function in every interval in which x varies, but solely a greatest or a least value in interval sufficiently small containing the value c.* Consequently a function may have several maximum and minimum values in any given interval

and *a minimum value may be actually greater than a maximum value.*

2. Theorem. *The only points where $f(x)$ can have extremum values are those at which the derivative is zero or does not exist.*

For if the derivative exists and is not zero, the function $f(x)$ is increasing or decreasing at the point according as the derivative is positive or negative and consequently acquires in the vicinity of that point values greater as well as smaller than that at the point and therefore $f(x)$ cannot be extremum at the point. Thus the condition is necessary (though not sufficient).

It may be remarked that *the points where $f'(x)$ is zero are those where the tangent to the curve $y=f(x)$ is parallel to the axis of x* like the points P and Q in the figure.

As an illustration, let us consider

$$f(x)=ax-x^2$$

we have $$f'(x)=a-2x\ ;$$

the derivative $f'(x)$ is zero for $x=a/2$ and it changes sign positive to negative when x increases from $a/2-\delta$ to $a/2+\delta$; therefore at first the function is increasing and then decreasing and consequently the function passes through a maximum value when $x=\frac{1}{2}a$, the maximum value being $\frac{1}{4}a^2$.

3. Distinction between Maxima and Minima. Suppose we are required to find the maximum and minimum values of a function $f(x)$ which possesses the first derivative. It has just been shown that the only values for which $f(x)$ is extremum are the roots of the equation $f'(x)=0$.

Let c be one of these roots, so that $f'(c)=0$. Then by Taylor's Theorem with remainder after two terms

$$f(c+h)-f(c)=\frac{h^2}{\lfloor 2}f''(c+\theta h), \qquad 0<\theta<1$$

$$=\frac{h^2}{\lfloor 2}\left[f''(c)\pm\epsilon\right]$$

where ϵ tends to zero as h tends to zero provided that $f''(x)$ is continuous at $x=c$. If $f''(c)$ is not equal to zero, then h can be so selected that the absolute value of ϵ is less than that of $f''(c)$ so that the expression on the right hand side keeps the same sign as that of $f''(c)$. *If $f''(c)$ is negative then $f(c+h)<f(c)$ and therefore $f(c)$ is a maximum; if $f''(c)$ is positive, then $f(c+h)>f(c)$ and therefore $f(c)$ is a minimum.*

But if $f''(c)$ is zero, then it requires further consideration. More generally, let us assume that the first derivative which is not zero is of order n. Then by Taylor's Theorem.

$$f(c+h)-f(c)=\frac{h^n}{\lfloor n}f^n(c+h\theta)=\frac{h^n}{\lfloor n}\quad f^n(c)\pm\epsilon$$

If n be even, the right-hand side keeps the same sign as that of $f^n(c)$ for sufficiently small values of h. Hence if $f''(c$ is positive, then $f(c+h>f(c)$ and $f(c)$ is a minimum ; and if $f''(c)$ is negative, then $f(c+h)<f''(c)$ and therefore $f(c)$ is a maximum.

If n be odd, then the right-hand side changes sign with h and consequently $f(c)$ is neither a maximum nor a minimum.

Hence we have the **following rule** : *For finding the turning values of a continuous function $f(x)$ in an interval where the derivative exists and is finite, we study the roots of $f'(x)=0$. Let c be one of these roots. We substitute $x=c$ in all the successive derivatives of $f(x)$ till we find one which is not zero for $x=c$. If that derivative is of odd order then there is neither muxima nor minima at $x=c$; but if that derivative is of even order then $f(e)$ is maximum if this derivativ e is negative and minimum if it is positive. In a similar manner we can treat the other roots of $f(x)=0$.*

Note. The student should note a few other properties of maxima and minima values of a continuous function.

1. Between two equal values of a function at least one maximum or one minimum must lie.

2. Between two maxima at least one minimum must lie ; and between two minima at least one maximum must lie. In other words maximum and minimum values must occur alternately.

3. In the immediate neighbourhood of a maximum or minimum value, the function has two values equal, one on each side.

Ex. Find the maximum and minimum values of the function $x^5-\frac{5}{2}x^4+5$.

Here $f(x)=x^5-\frac{5}{2}x^4+5$

$\therefore\ f'(x)=5x^4-10x^3$ and $f''(x)=20x^3-30x^2$.

The roots of the equation $f'(x)=0$, or of $5x^3(x-2)=0$ are 2 and 0.

Substituting $x=2$ in $f''(x)$, we find that it is equal to 40, so that $f''(x)$ is positive.

Therefore $f(x)$, is minimum for $x=2$, the minimum value being -3.

For $x=0$, $f''(x)=0$, and differentiating further, we find

$$f'''(x)=60x^2-60x$$

$$f^{iv}(x)=120x-60$$

and for $x=0$, $f'''(x)=0$, whereas $f^{iv}(x)=-60$.

Therefore for $x=0$, $f(x)$ is maximum, the maximum value being 5.

4. Rule when f′(x) exists in the vicinity of x=c but may not exist at x=c. Let $f(x)$ be a function having a derivative $f'(x)$ in the vicinity of the point $x=c$ (the derivative at $x=c$ might or might not exist); suppose further that $f'(x)$ has, in the vicinity of the point $x=c$ a unique sign for $x<c$ and a unique sign for $x>c$. Let x pass through the value c while increasing. If $f'(x)$ changes sign from negative to positive, then $f(x)$ is minimum for $x=c$; and if $f'(x)$ changes sign from positive to negative, then $f(x)$ is maximum for $x=c$. And if $f'(x)$ does not change sign, then $f(x)$ is neither maximum nor minimum at $x=c$.

For we know that $f(x)$ is increasing or decreasing according as the sign of $f'(x)$ is positive or negative. In the first case when $f'(x)$ changes sign from negative to positive, then $f(x)$ decreases and attains the value $f(c)$ and increases afterwards so that $f(c)$ is a minimum. In the second case when $f'(x)$

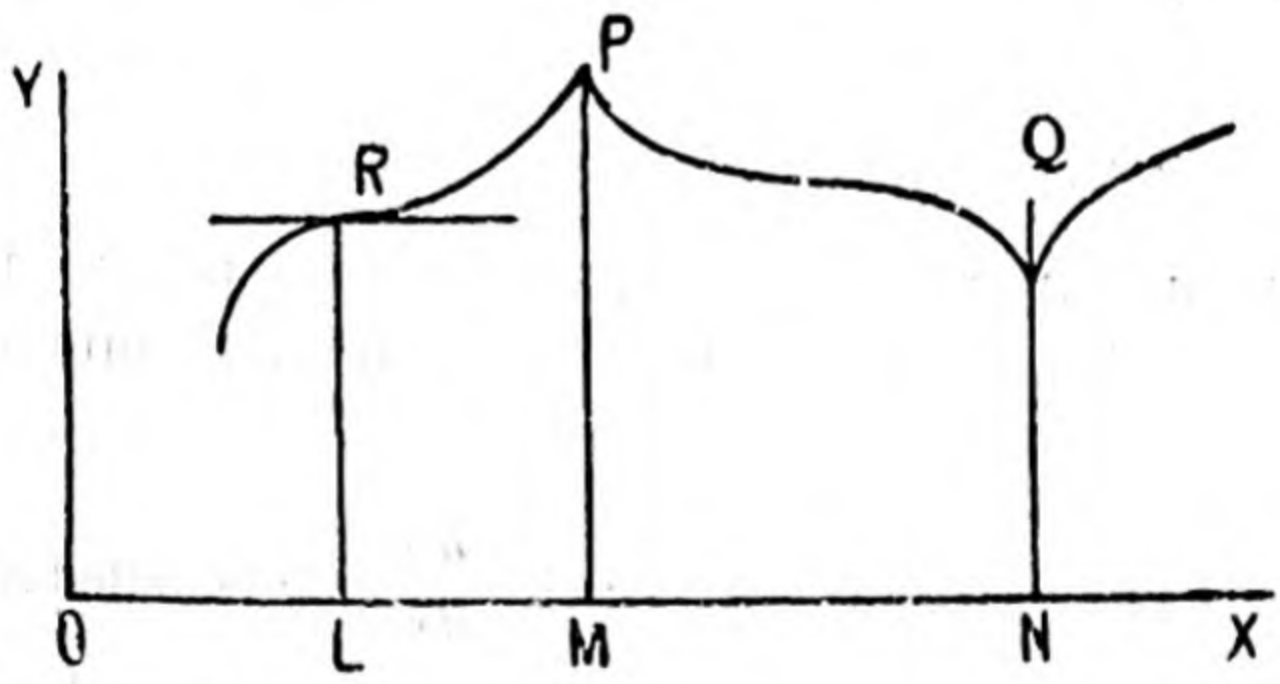
changes sign from positive to negative, the function $f(x)$ increases and attains the value $f(c)$ and then decreases so that $f(c)$ is a maximum. In the last case, when $f'(x)$ does not change sign, $f(x)$ continues to increase or continues to decrease when x passes through the value c, so that $f'(c)$ is neither a maximum nor a minimum.

It may be remarked that sometimes this rule gives the result more easily than any other.

Ex. 1. If $f'(x)=(x-a)^{2n}\,(x-b)^{2m+1}$, m and n being positive integers, show that $x=a$ gives neither a maximum nor a minimum of $f(x)$, but $x=b$ gives a minimum.

When x passes through the value a, $f'(x)$ keeps the same sign in a small vicinity, so that $x=a$ gives neither a maximum nor a minimum. When x passes through b, $f'(x)$ changes from negative to positive and therefore $x=b$ makes $f(x)$, a minimum.

Ex. 2. Consider the function $f(x)$ whose graphs is as shown in the figure. Then (*i*) at the point Q the ordinate NQ is



minimum even though the derivative at Q does not exist ; (*ii*) at the point P, the ordinate MP is maximum even though the derivative at P does not exist ; and (*iii*) at the point R, the ordinate LR is neither maximum non minimum even though $f'(x)=0$ at the point R. because $f'(x)$ does not change sign as x passes through R. This illustrates the fact that maximium and minimum values, may occur at points where the derivative $f'(x)$ does not exist and these may not occur at points where $f(x)=0$.

Ex. 3. Find the stationary points of the function $x^5-5x^4+5x^3+1$ and examine for which of them the function is a maximum.

Here, $f(x)=x^5-5x^4+5x^3+1$

$\therefore$ $f'(x)=5(x^4-4x^3+3x^2)$

$f''(x)=5(4x^3-12x^2+6x)$.

Putting $f'(x)=5x^2(x^2-4x+3)=0$, we get

$$x=3,\ x=1,\ x=0. \qquad \ldots(1)$$

These give the *stationary points of the function* : substituting these values in $f''(x)$, we have

(*i*) $f''(3)=5(4\times27-12\times9+6\times3)=+90$

$\therefore$ $x=3$ makes $f(x)$ *a minimum.*

(*ii*) $f''(1)=5(4\times1^3-12\times1^2+6\times1)=-10$

$\therefore$ $x=1$ makes $f(x)$ *maximum.*

(*iii*) $f''(0)=0$. This requires further investigation

$f'''(x)=5(12x^2-24x+6)$

$f''''(x)=30$.

As the first derivative which is not zero is of odd order, therefore, the function is neither maximum nor minimum for $x=0$.

Note. The value of x at which $\frac{dy}{dx}=0$, is called a **critical value.** A function y cannot have an extreme value except at a critical point but the converse is not true.

Ex. 4. At a variable point P of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, a normal is drawn to the ellipse. Find the maximum distance of the normal from the centre of the ellipse.

The normal at the point P ($a\cos\phi$, $b\sin\phi$) to the ellipse is

$$a\sec\phi.\ x-b\ \mathrm{cosec}\ \phi.\ y=a^2-b^2,$$

The length of the perpendicular from (0, 0) on the normal is

$$\frac{(a^2-b^2)}{\sqrt{a^2\sec^2\phi+b^2\operatorname{cosec}^2\phi}} \qquad \ldots\ldots(1)$$

The perpendicular will be maximum when

$u=a^2\sec^2\phi+b^2\operatorname{cosec}^2\phi$ is *minimum*.

$$\therefore \quad \frac{du}{d\phi}=2a^2\sec^2\phi\tan\phi-2b^2\operatorname{cosec}^2\phi\cot\phi=0$$

Whence $\dfrac{\sec^2\phi\tan\phi}{\operatorname{cosec}\phi\cot\phi}=\dfrac{b^2}{a^2}$ or $\tan^4\phi=\dfrac{b^2}{a^2}$

i. e., $\tan^2\phi=\dfrac{b}{a}$.

Now $\dfrac{d^2u}{d\phi^2}=2a^2\sec^4\phi+4a^2\sec^2\phi\tan^2\phi+2b^2\operatorname{cosec}^4\phi$ $+4b^2\operatorname{cosec}^2\phi\cot^2\phi$, which is positive so that u is minimum.

Hence the perpendicular is maximum when $\tan^2\phi=\dfrac{b}{a}$.

∴ The maximum length of the central perpendicular is from (1) equal to

$$\frac{a^2-b^2}{\sqrt{a^2\left(1+\frac{b}{a}\right)+b^2\left(1+\frac{a}{b}\right)}}=\frac{a^2-b^2}{\sqrt{a(a+b)+b(a+b)}}$$

EXAMPLES XVII

Examine the following functions for maximum and minimum values :—

1. $y=x^3-18x^2+96x$.
2. $y=2x^3-24x^2+42x+10$.
3. $y=(x-3)^5(x+1)^4$.
4. Find the maximum value of x^3-3x+4 and explain why the function can be made to assume values greater than the maximum.

5. How do you explain the fact that though
$$y=2x^3-9x^2+12x+1$$
can be made as large as we please by taking x to be large enough, the function has a maximum value when $x=1$?

6. Show that $\dfrac{1-x+x^2}{1+x-x^2}$ has only one minimum at $x=\frac{1}{2}$ and that it has no maximum.

Show that the following functions have *neither maximum nor minimum values* :—

7. (*i*) x^3-3x^2+6x-5. (*ii*) $\dfrac{ax+b}{cx+d}$ where $ad-bc\neq 0$.

8. ax^2+bx+c where a, b, c are positive.

9. $(x-1)^3+2x$.

10. Find the extremum values of x
$$y=10x^6-12x^5+15x^4-20x^2+18.$$

11. Investigate the maxima and minima of
$$12x^5-15x^4+40x^3+6.$$

12. Prove that $x^5-5x^4+5x^3-10$ has a maximum for $x=1$, a minimum for $x=3$; and for $x=0$ it has neither a maximum nor a minimum.

13. Find maximum and minimum values of
$$3x^5-25x^3+60x.$$

14. Find for what values of x the following expression is a maximum and a minimum respectively :
$$2x^3-21x^2+36x-20.$$

15. Investigate the maxima and minima of
$$(x-2)^6(x-3)^5.$$

16. Find the maxima and minima of $\dfrac{x^3+x}{x^4-x^2+1}$.

17. Show that $f(x)=\dfrac{(x+3)^3}{(x+2)^2}$ is minimum for $x=0$.

18. Find the function $\dfrac{ax^2+2bx+c}{Ax^2+2Bx+c}$ which has turning values -1 and $\frac{3}{5}$ for $x=0$ and $x=2$ respectively and which is equal to 1 for $x=1$.

19. Prove that $x^{\frac{1}{x}}$ has a maximum for $x=e$.

20. Prove that the minimum value of

$$\frac{a^2}{\cos^2\theta}+\frac{b^2}{\sin^2\theta} \text{ is } (a+b)^2$$

21. Show that a maximum or minimum of $f(x)$ is a minimum or maximum of $\frac{1}{f(x)}$.

22. Prove that the function

$$m_1(x-x_1)^2+m_2(x-x_2)^2+\ldots\ldots+m_n(x-x_n)^2$$

is a minimum when

$$x=\frac{m_1x_1+m_2x_2+\ldots\ldots+m_nx_n}{m_1+m_2+\ldots\ldots+m_n}.$$

23. The equation of the path of a projectile being $y=x \tan \alpha - x^2/4h \cos^2 \alpha$, find the value of x where y is maximum. Find also the maximum value of y.

24. A man is 3 miles away from the nearest point P of a straight bank of a lake and he wishes to reach a place 6 miles from P along the bank. He can row at 4 miles an hour while he can walk 5 miles per hour. Find the least time in which he can reach his destination and the place where he should land.

25. Divide a given number a into two parts such that the product of one part with the cube of the other is a maximum.

26. If $a \sec x+b \sec y=c$, show that for $x=y$, $a \cos x+b \cos y$ is a minimum, a, b, c being positive and x and y each acute.

27. Show that the height of the cylinder of maximum volume inscribed in a right circular cone is one third that of the cone.

28. Show that the height of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius r is $2r/\sqrt{3}$.

Show that when the curved surface is a maximum, the height is $r/\sqrt{2}$ and that when the whole surface is maximum, that surface is

$$\pi r^2(\sqrt{5}+1).$$

29. The sum of the surfaces of a sphere and a cube is given Show that when the sum of the volumes is least, the diameter of the sphere is equal to the edge of the cube.

30. Discuss the maximum and minimum values of

(*i*) $\sin x \cos^3 x$ (*ii*) $\sin^n x . \sin nx$ (*iii*) $a \sin^3\theta \cos^4\theta$ (*iv*) $a \sin^2\theta + b\cos^2\theta$.

31. Find the maximum value of $(x-a)^2 (x-b)$,

(*i*) when $a>b$, (*ii*) when $a<b$.

What happens if $a=b$? Illustrate your answers by tracing the curve $y=(x-a)^2(x-b)$ in the three different cases.

32. Show that the function $4 \cos\theta + \cos 2\theta$ is maximum or minimum when $\cos\theta$ is maximum or minimum.

33. On a given line OX, find a point P such that the sum of the distances of two given points A and B from P shall be minimum.

34. If the sum of the edges of a rectangular parallelopiped is l and the sum of the areas of the faces is $l^2/25$, show that when the excess of the volume of the parallelopiped over that of a cube whose edge is its smallest edge, is as great as possible, the smallest edge must be $l/20$ and find the lengths of the other edges.

35. Prove that in any triangle the maximum value of

(*i*) cos A cos B cos C is $\frac{1}{8}$. (*ii*) sin A sin B sin C is $3\sqrt{3}/8$.

36. If P is any point within a triangle ABC such that $AP^2+BP^2+CP^2$ is a minimum, show that P coincides with the centroid.

37. If $x+y+z+w+\ldots\ldots$ be constant, show that $xyzw \ldots\ldots$ will have its maximum value where $x=y=z=w=\ldots$

Show that the maximum value of xy^2z^3 when $x+y+z=a$ is $a^6/2^4.3^3$.

5. We have already seen (Art. 3, Note) that a function is a maximum (or minimum) when it lies between *two equal adjacent values* of the function and the position of maximum (or minimum) value is the position of coincidence of these two equal values, which are each less or each greater than the turning value according as the turning value is a maximum or minimum.

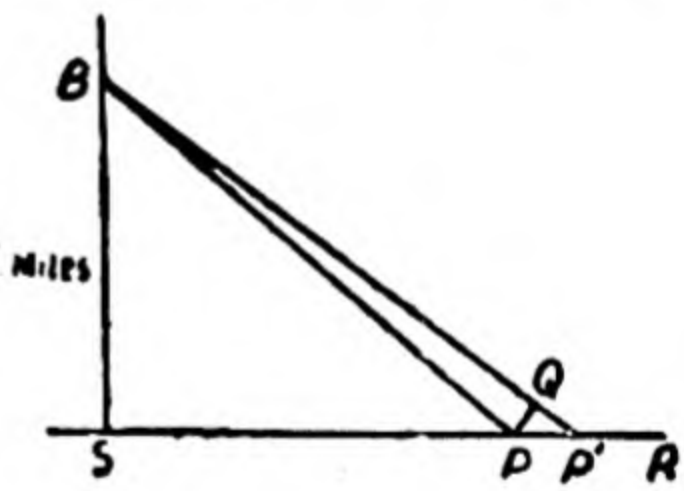
It should, however, be noted that the equal adjacent values are not necessarily at equal distances from the position of the maximum (or minimum) but *somewhere* in the neighbourhood.

This follows at once from the continuity of the function and the unbroken continuous curve which it represents. For every point on the left of P (or Q) [see fig. of Art. 1) there is a corresponding point on the right. Hence corresponding to an ordinate on the left, there must be an equal ordinate on the right in the neighbourhood of P, (or Q) since the change in the value of the function is also continuous.

As an illustration of this principle, we shall solve the following example.

Example. A person being in a boat 3 miles from the nearest point of the beach, wishes to reach in the shortest time a place 5 miles from that point along the shore; supposing he can walk 5 miles an hour but pull only at the rate of 4 miles an hour, it is required to find the place where he must land.

Let B be the position of the boat and SR the beach ; SB=3 miles, SR=5 miles, BPR and BP'R are any two paths. Cut off from BP' a length BQ=BP, so that BP'−BP =QP' and ∠ Q=a rt. angle ultimately. From the consideration that the time in the two adjacent paths BPR and BP'R is the same if a maximum or a minimum lies between them, we get



$$\frac{BP}{4}+\frac{PR}{5}=\frac{BP'}{4}+\frac{P'R}{5}.$$ [P and P' being ultimately coincident points.]

$$\therefore\quad \frac{BP'-BP}{4}=\frac{PR-P'R}{5}$$

$$\text{i. e.,}\quad \frac{QP'}{4}=\frac{PP'}{5}\quad \text{or}\quad \cos P'=\frac{4}{5}\ ;$$

$$\text{Hence}\quad \tan P'=\frac{3}{4}=\frac{BS}{SP}=\frac{3}{SP}$$

$$\therefore\quad SP=4.$$

∴ The man must land at a point one mile from R, the place to be reached.

EXAMPLES XVIII

1. Show that the greatest and the least straight lines which can be drawn from any point within an ellipse to the curve are normal to the curve.

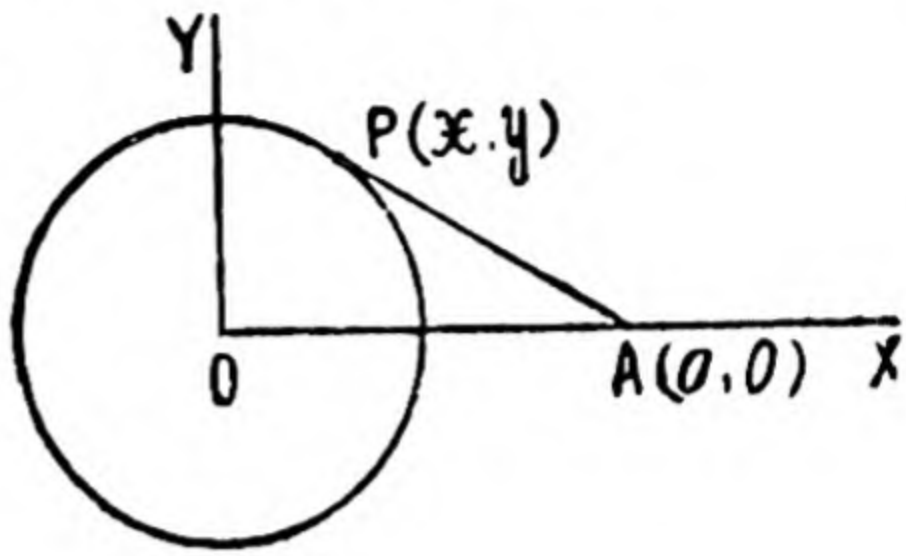
2. Show that a triangle of maximum area inscribed in any oval curve is such that the tangent at each angular point is parallel to the opposite side.

3. Show that the sides of a triangle of minimum area circumscribing any oval curve are bisected at the points of contact. Prove further that if the oval be an ellipse, the centre of gravity of such a triangle coincides with the centre of the ellipse.

4. In a oval curve the maximum or minimum chord which is normal at one end is either a radius of curvature at that end or normal at both ends.

6. **Greatest and Least Values.** Sometimes there arises the necessity of finding the greatest or the least value of a function $f(x)$ of the variable x when x varies between two given limits a and b. For solving this problem, it is necessary to study the changes of the function $f(x)$ by using its derivative. The simplest case is when the derivative remains finite and keeps the same sign; if it is positive throughout, then $f(x)$ is an increasing function throughout, as x varies from a to b, and consequently the greatest and least values of $f(x)$ are $f(b)$ and $f(a)$ respectively; while if the derivative is negative throughout, then $f(x)$ is a decreasing fuuction and the greatest and the least values of $f(x)$ are $f(a)$ and $f(b)$ respectively. When the derivative does not keep the same sign and the function has several maxima and minima, then the greatest of the maxima values and $f(a)$ and $f(b)$, is the greatest value of the function. Similarly the least of the minimum values and $f(a)$ and $f(b)$ is the least value of the function.

Ex. 1. Find the greatest and the least distances of a given point A from the circumference of a given circle of radius r.



Take axes as in the figure. We are required to find the greatest and the least value of AP.

Let $u=\text{AP}^2=(x-a)^2+y^2$.

But the equation of the circle being $x^2+y^2=r^2$,

we get $y^2=r^2-x^2$;

therefore $u=r^2+a^2-2ax$.

Also it is clear that x varies only from $-r$ to r.

$\frac{du}{dx}=-2a$, which is negative throughout.

Therefore the greatest and the least values of u correspond to $x=-r$ and $x=r$ respectively. Hence the greatest and the least value of u are $(r+a)^2$ and $(r-a)^2$ and therefore those of AP are $(r+a)$ and $(r-a)$.

Ex. 2. Find the greatest and the least values of $f(x)=x^3-18x^2+96x$ in the interval (0, 9) and draw the graph of the function in this interval.

Here $f'(x)=3x^2-36x+96=3(x-4)(x-8)$;

so that $f'(x)=0$ for $x=4$ and $x=8$.

For $x=4-\epsilon$, $f'(x)$ is positive; and for $x=4+\epsilon$., $f'(x)$ is negative.

Therefore for $x=4$, $f(x)$ is minimum, its value being 160.

Also for $x=8-\epsilon$, $f'(x)$ is negative and for $x=8+\epsilon$., $f'(x)$ is positive.

Therefore for $x=8$, $f'(x)$ is minimum, its value being 128.

Now as x varies from 0 to 4, $f(x)$ is an increasing functions; as x varies from 4 to 8, $f(x)$ is a decreasing function; and as x varies from 8 to 9, $f'(x)$ is positive and therefore $f(x)$ is an increasing function, its value for $x=9$ being 135,

Therefore the function starts from 0, increases and attains the value 160 for $x=4$; then it decreases and attains the value 128 for $x=8$ and then increases and attains the value 135 for $x=9$.

Therefore the least value is 0 and the greatest value is 160.

The graph of the function can now be easily drawn.

EXAMPLES XIX

1. Find the maxima and minima as well as the least and the greatest values of the function $y=x^3-12x^2+45x$ in the interval (0, 7) and draw a sketch of the function in the same interval.

2. Find the least and the greatest values of $x^3-9x^2+24x+1$ in the interval (0, 5).

3. Show how to find the maximum and minimum values of a function of a single variable from the variation of its derivative and apply the method to prove that $\frac{1}{8}$ $(35\cos^4x-30\cos^2x+3)$ ranges in values between 1 and $-\frac{3}{7}$ and has also $\frac{3}{8}$ as a maximum value. (M. T.)

4. Show that the minimum value of the function

$$\sec^4\theta-6\sec^2\theta+8 \text{ is } -1. \qquad \text{(1919 Trip.)}$$

5. Find the equation whose roots give the values of x for which the function

$$f(x)=ae^{-x^2}-be^{-(x-c)^2}$$

is a maximum or minimum, and show that if x_1 is a root, then

$$f(x_1)=\frac{ac}{c-x_1}e^{-x_1^2}. \qquad \text{(Trip.)}$$

7. Implicit Functions. Consider y as a function of x given by the equation

$$f(x, y)=0. \qquad (i)$$

Differentiating this equation, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0. \qquad (ii)$$

The condition that y may be maximum or minimum is that $\frac{dy}{dx}$ shall be zero; consequently we have

$$\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}=0 \qquad (iii)$$

Let (x_1, y_1), (x_2, y_2)...be the solutions common to the equations (i) and (ii) ; the values of x which make y extremum are included in the values x_1, x_2...

In order to distinguish between maxima and minima, let us differentiate equation (ii). We get

$$\frac{\partial^2 f}{\partial x^2}+2\frac{\partial^2 f}{\partial x\,\partial y}\frac{dy}{dx}+\frac{\partial^2 f}{\partial y}\left(\frac{dy}{dx}\right)^2+\frac{\partial f}{\partial y}\frac{d^2y}{dx^2}=0 \qquad (iv)$$

and since $\frac{dy}{dx}$ is zero for the values under consideration, this equation gives

$$\frac{d^2y}{dx^2}=-\frac{\frac{\partial^2 f}{\partial x^2}}{\frac{\partial f}{\partial y}};$$

the sign of this expression indicates a maximum or a minimum. If $\frac{d^2y}{dx^2}$ reduces to zero, we follow a method already indicated.

Remarks. It may happen that some of the solutions common to (i) and (iii) also satisfy simultaneously the equations

$$\frac{\partial f}{\partial x}=0,\quad \frac{\partial f}{\partial y}=0;$$

for these particular values of x and y, $\frac{dy}{dx}$ is not given by equation (ii) and consequently y may or may not have a maximum or a minimum at such points.

Ex. 1. Let $f(x, y)=y^3-3x^2y+x^3-3=0.$ (i)

Equating $\frac{\partial f}{\partial x}$ to zero, we have $3x^2-6xy=0,$ (ii)

from which we get $x=0$ or $x=2y.$

Putting $x=0$ in (i) we get

$$y^3-3=0, \text{ or } y=\sqrt[3]{3}.$$

Substituting the second value, we get $y^3+1=0,$
so that $y=-1$ and consequently $x=-2.$

The solutions of (i) and (ii) are therefore

$$\begin{matrix} x=0 \\ y=\sqrt[3]{3} \end{matrix} \text{ and } \begin{matrix} x=-2 \\ y=-1. \end{matrix}$$

Also $\dfrac{\partial f}{\partial y}=3y^2-3x^2$ and none of the above solutiions satisfies $\dfrac{\partial f}{\partial y}=0$.

Now for the values under consideration

$$\frac{d^2y}{dx^2}=-\frac{\dfrac{\partial^2 f}{\partial x^2}}{\dfrac{\partial f}{\partial y}}=-\frac{6x-6y}{3y^2-3x^2}=+\frac{2}{x+y}.$$

For $x=0$, $y=\sqrt[3]{3}$, this is $\dfrac{2}{\sqrt[3]{3}}$, positive and therefore $y=\sqrt[3]{3}$ is a minimum.

For $x=-2$, $y=-1$, this is $-\frac{2}{3}$ and therefore $y=-1$ is a maximum.

Ex. 2. Consider $f(x, y)=x^3+y^3-3axy=0$. .....(*i*)

Here $\dfrac{\partial f}{\partial x}=-3ay+3x^2$. .....(*ii*)

Eliminating y between $\dfrac{\partial f}{\partial x}=0$ and $f(x, y)=0$,

we get $$x^6-2a^3x^3=0$$

so that $x=0$ and $x=\sqrt[3]{2a}$,

and the corresponding values of y are

$$y=0 \text{ and } y=\sqrt[3]{4a}.$$

The solutions of (i) and (ii) therefore are

$$\begin{matrix} x=0 \\ y=0 \end{matrix} \text{ and } \begin{matrix} x=\sqrt[3]{2a}. \\ y=\sqrt[3]{4a}. \end{matrix}$$

Also $\dfrac{\partial f}{\partial y}=3y^2-3ax$ and the solution $x=0$, $y=0$ satisfies $\dfrac{\partial f}{\partial y}=0$, while $x=\sqrt[3]{2a}$ and $y=\sqrt[3]{4a}$ does not.

Now for the second solution,

$$\frac{d^2y}{dx^2}=-\frac{\dfrac{\partial f}{\partial x^2}}{\dfrac{\partial f}{\partial y}}=-\frac{-2ax}{y^2-ax}=-\frac{2}{a};$$

therefore for $x=\sqrt[3]{2a}$, $y=\sqrt[3]{4a}$, y is a maximum.

For the second solution $x=0$, $y=0$, $\frac{dy}{dx}$ is not given by the equation $\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0$,

$$\text{or } (3x^2-3ay)+(3y^2-3ax)\frac{dy}{dx}=0$$

for $\frac{dy}{dx}=\frac{ay-x^2}{y^2-ax}$ which takes the form $\frac{0}{0}$ and also $\frac{d^2y}{dx^2}$ is of the same form. But further differentiations give

$$(y^2-ax)\frac{d^2y}{dx^2}+2y\left(\frac{dy}{dx}\right)^2-2a\frac{ay}{dx}+2x=0 \qquad \text{(A)}$$

and $$(y^2-ax)\frac{d^3y}{dx^3}+3\left(2y\frac{dy}{dx}-a\right)\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^3+2=0 \qquad \text{(B)}$$

and putting $x=0$, $y=0$ in these equations, we get $\frac{dy}{dx}=0$ from (A) and consequently $\frac{d^2y}{dx^2}=\frac{2}{3a}$ from (B). Therefore for $x=0$, $y=0$, y is a minimum.

8. Let $u=f(x, y)$ be a function of two variables x and y which are connected by the relation $F(x, y)=0$, ...(*i*)
so that u is really a function of only one variable, say x.

For u to be maximum or minimum

$$\frac{du}{dx}=0.$$

$$\therefore \frac{du}{dx}=\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y}\frac{dy}{dx}=0 \qquad \text{.....(ii)}$$

where $\frac{dy}{dx}$ is given by differentiating the equation (i)

$$\frac{\partial F}{\partial x}+\frac{\partial F}{\partial y}\frac{dy}{dx}=0.$$

Eliminating $\frac{dy}{dx}$ between (i) and (ii), we get

$$\frac{\partial f}{\partial x}\frac{\partial F}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial F}{\partial x}=0. \qquad (iv)$$

This equation combined with (i) gives the pairs of values of x and y, for which $f(x, y)$ may be maximum or minimum.

For further discussion, the higher derivatives of x have to be found and the process already indicated is to be applied.

Remark. This method is perfectly general and can be applied to a function u of n variables, which are themselves connected by $(n-1)$ relations.

Ex. 1. $u=\frac{a^2}{x^2}+\frac{b^2}{y^2}$ where $x+y=c$.

$$\frac{du}{dx}=-\frac{2a^2}{x^3}-\frac{2b^2}{y^3}\frac{dy}{dx}$$

and $1+\frac{dy}{dx}=0$ so that $\frac{dy}{dx}=-1$.

$$\therefore \quad \frac{du}{dx}=-\frac{2a^2}{x^3}+\frac{2b^2}{y^3}=0$$

or $$\frac{a^2}{x^3}=\frac{b^2}{y^3} \quad \therefore \quad \frac{a^{\frac{2}{3}}}{x}=\frac{b^{\frac{2}{3}}}{y}=\frac{a^{\frac{2}{3}}+b^{\frac{2}{3}}}{c}$$

so that $x=\frac{ca^{\frac{2}{3}}}{a^{\frac{2}{3}}+b^{\frac{2}{3}}}$, $\quad y=\frac{cb^{\frac{2}{3}}}{a^{\frac{2}{3}}+b^{\frac{2}{3}}}$

$$\frac{d^2u}{dx^2}=\frac{6x^2}{x^4}-\frac{6b^2}{y^4}\frac{dy}{dx}$$

$$=\frac{6a^2}{x^4}+\frac{6b^2}{y^4}$$

which is positive. Therefore we find that for

$$x=\frac{ca^{\frac{2}{3}}}{a^{\frac{2}{3}}+b^{\frac{2}{3}}},\quad y=\frac{cb^{\frac{2}{3}}}{a^{\frac{2}{3}}+b^{\frac{2}{3}}}\ u \text{ is minimum, the}$$

minimum value being $\frac{1}{c^2}\left(a^{\frac{2}{3}}+b^{\frac{2}{3}}\right)^3$.

Ex. 2. Find the maximum and minimum values of

$$u=(x-x_1)^2+(y-y_1)^2$$

where $$ax+by+c=0. \qquad \ldots\ldots(i)$$

Interpret the result geometrically.

Here $\frac{du}{dx}=2(x-x_1)+2(y-y)_1\frac{dy}{dx}$

and $a+b\frac{dy}{dx}=0$ or $\frac{dy}{dx}=-\frac{a}{b}$.

Therefore $\frac{du}{dx}=2(x-x_1)-2\frac{a}{b}(y-y_1)$.

$\frac{du}{dx}=0$ gives $(x-x_1)=\frac{a}{b}(y-y_1)$.

Solving this with (i), we find that

$$y=\frac{a^2y_1-abx_1-bc}{a^2+b^2}$$

and therefore, $y-y_1=-\frac{b(ax_1+by_1+c)}{a^2+b^2}$.

$$\text{Also}\frac{d^2u}{dx^2}=2-2\frac{a}{b}\frac{dy}{dx}$$

$$=2+2\frac{a^2}{b^2}$$

which is positive. Therefore u is minimum

$$u=(x-x_1)^2+(y-y_1)^2$$

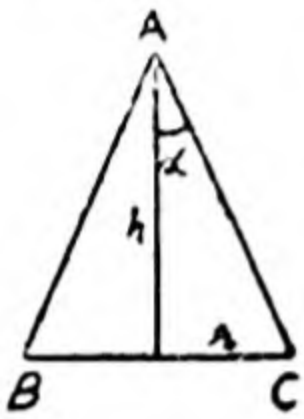
$$=\frac{a^2}{b^2}(y-y_1)^2+(y-y_1)^2$$

$$=(y-y_1)^2\frac{a^2+b^2}{b^2}=\frac{(ax_1+by_1+c_1)^2}{a^2+b^2}.$$

Geometrically, the shortest or perpendicular distance of any point (x_1, y_1) from the line $ax+by+c=0$ is

$$\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}.$$

Ex. 3. Show that the semi-vertical angle of the right cone of given surface and maximum volume is $\sin^{-1}\frac{1}{3}$.



Volume of cone $=\frac{1}{3}\pi r^2 h$.

Curved surface $=\pi\sqrt{r^2+h^2}=\pi r^2$ cosec α.

Surface including plane base $=\pi r\sqrt{r^2+h^2}+\pi r^2$

We have

$$v=\tfrac{1}{3}\pi r^2 h \qquad \ldots(1)$$

Where r and h are connected by the relation

$$\pi r\sqrt{r^2+h^2}+\pi r^2=k^2 \qquad \ldots(2)$$

Differentiating (1)

$$\frac{dv}{dr}=\tfrac{2}{3}\pi rh+\tfrac{1}{3}\pi r^2\frac{dh}{dr} \qquad \ldots(3)$$

Differentiating (2)

$$0=\pi\sqrt{r^2+h^2}+\pi r.\frac{r+h\dfrac{dh}{dr}}{\sqrt{r^2+h^2}}+2\pi r,$$

i.e., $$0=(r^2+h^2)+r^2+rh\frac{dh}{dr}+2r\sqrt{r^2+h^2}$$

whence

$$\frac{dh}{dr}=-\frac{2r^2+h^2+2r\sqrt{r^2+h^2}.}{rh} \qquad \ldots(4)$$

Substituting this value in (3)

$$\frac{dv}{dr}=\frac{1}{3}\pi r\left(2h+r\times\frac{-2r^2+h^2+2r\sqrt{r^2+h^2}).}{rh}\right) \qquad \ldots(5)$$

$$=\tfrac{1}{3}\pi\frac{r}{h}\ (h^2-2r^2-2r\sqrt{r^2+h^2})$$

Putting $\frac{dv}{dr}=0$, we have,

$h^2-2r^2-2r\sqrt{r^2+h^2}=0$

or $h^4+4r^4-4h^2r^2=4r^2(r^2+h^2)=4r^4+4r^2h^2$

or $\mathbf{h^4=8h^2r^2}$, *i.e.*, $\dfrac{\mathbf{r}}{\mathbf{h}}=\dfrac{\mathbf{1}}{\mathbf{2\sqrt{2}}}$; $\tan\alpha=\dfrac{\mathbf{1}}{\mathbf{2\sqrt{2}}}$

$\therefore$ $\sin\alpha=\frac{1}{3}$.

Hence an extreme value of volume is given by $\sin\alpha=\frac{1}{3}$...(6)

We shall show in the following notes that $\alpha=\sin^{-1}\frac{1}{3}$ corresponds to the maximum volume.

Note 1. We observe that α can only vary from 0 to $\frac{\pi}{2}$ and when $\alpha=0$, the volume is evidently minimum, therefore $\alpha=\sin^{-1}\frac{1}{3}$ corresponds to the maximum volume.

Note 2. That $\alpha=\sin^{-1}\frac{1}{3}$ corresponds to the maximum volume and not to the minimum can also be shown in the following manner :—

From (5), we have

$$\frac{dv}{dr}=\frac{1}{3}\pi\tan\alpha.h^2(1-2\tan^2\alpha-2\tan\alpha\sec\alpha)$$

$$=\tfrac{1}{3}\pi\tan\alpha.h^2(3-2\sec^2\alpha-2\tan\alpha\sec\alpha)$$

$$=\frac{1}{3}\pi\tan\alpha.h^2\left\{\left(\frac{9}{4}-2\sec^2\alpha\right)+\left(\frac{3}{4}-2\tan\alpha\sec\alpha\right)\right\}$$

Now $\sin\alpha=\frac{1}{3}$,

$\therefore$ $\sec\alpha=\dfrac{3}{2\sqrt{2}}$, $\tan\alpha=\dfrac{1}{2\sqrt{2}}$.

$\therefore$ As $\sin\alpha$ changes from a value slightly less than $\frac{1}{3}$ to a value slightly greater than $\frac{1}{3}$, it is evident the two expressions within the brackets change from positive to negative and the factor outside $\frac{1}{3}\pi\tan\alpha^2.h$ remains positive, therefore $\dfrac{dv}{dr}$ changes from *plus* to *minus*.

$\therefore$ v (volume) is a maximum and not a minimum for $\alpha=\sin^{-1}\frac{1}{3}$.

Note 3. In geometrical problems it is always preferable to discriminate between maximum and minimum in the manner given in Note 1 than in Note 2.

GIRLS COL

Ex. 4. Find the maximum and minimum values of x^2+y^2, where $ax^2+2hxy+by^2=1$. (*i*)

Here regarding x as the independent variable, x^2+y^2 is a function of x only, say $\phi(x)$. Then

$$\frac{1}{2}\ \phi'(x)=x+y\frac{dy}{dx}.$$

Now $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are given by differentiating (*i*) twice. In fact we have

$$\frac{dy}{dx}=-\frac{ax+hy}{hx+by} \text{ and } \frac{d^2y}{dx^2}=\frac{h^2-ab}{(hx+by)^3}.$$

Therefore $\phi'(x)=0$ gives

$$x-y\ \frac{ax+hy}{hx+by}=0,$$

or
$$\frac{ax+hy}{x}=\frac{hx+by}{y}\text{.....(A)}$$

or
$$\frac{x(ax+hy)}{x^2}=\frac{y(hx+by)}{y^2}=\frac{ax^2+2hxy+by^2}{x^2+y^2}$$

$$=\frac{1}{r^2}, \text{ where } r^2=x^2+y^2.$$

Therefore $ax+hy=\frac{x}{r^2}$ and $hx+by=\frac{y}{r^2}$.

Eliminating x and y between these two equations, we get

$$\left(\frac{1}{r^2}-a\right)\left(\frac{1}{r^2}-b\right)-h^2=0$$

or $\frac{1}{r^4}-\frac{1}{r^2}(a+b)+ab-h^2=0$, which gives the turn-values of $\phi(x)$ or r^2.

Solving, $\frac{1}{r^2}=\frac{a+b\pm d}{2}$, ...(*i*)

where $d^2=(a-b)^2+4h^2$.

Now $\frac{1}{2}\phi''(x)=1+y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2$

$$=1+y\frac{h^2-ab}{(hx+by)^3}+\frac{x^2}{y^2}$$

$$=\frac{x^2+y^2}{y^2}+\frac{y(h^2-ab)r^6}{y^3}$$

$$=\frac{r^6}{y^2}\left[\frac{1}{r^4}+h^2-ab\right]$$

$$=\frac{r^6}{y^2}\left[\frac{(a+b\pm d)^2}{4}+h^2-ab\right]$$

$$=\frac{r^6}{y^2}\left[\frac{2a^2\pm 2(a+b)d}{4}\right]$$

Therefore $\phi''(x)=\frac{r^6 d}{y^2}[d\pm(a+b)]$.

Two cases arise :

(*a*) Let $ab>h^2$. In this case $a+b>d$ and therefore both the values of r^2 from (*i*) are positive and admissible so that $\phi(x)$ has two values, namely

$$\frac{2}{a+b-d} \text{ and } \frac{2}{a+b+d}.$$

When $\phi(x)=\frac{2}{a+b-d}$, $\phi''(x)$ is negative and this is therefore its maximum value.

When $\phi(x)=\frac{2}{a+b+d}$, $\phi''(x)$ is positive and this is therefore its minimum value.

(*b*) Let $ab<h^2$. In this case $a+b<d$ and therefore from (*i*) one value of r^2 is positive and the other is negative namely $\frac{2}{a+b-d}$ and is therefore an inadmissible value, so that $\phi(x)$ has only one value, $\frac{2}{a+b+d}$.

When $\phi(x)=\frac{2}{a+b+d}$, $\phi''(x)$ is positive and therefore this is its minimum value.

Geometrically, the lengths of the axes of the conic $ax^2+2hxy+by^2=1$ are given by the equation

$$\left(a-\frac{1}{r^2}\right)\left(b-\frac{1}{r^2}\right)=h^2,$$

where r is either semi-axis.

From A we get $h(y^2-x^2)+(a-b)xy=0$

i.e., $h\left(\frac{y^2}{x^2}-1\right)+(a-b)\frac{y}{x}=0,$

i.e., $h\tan^2\theta+(a-b)\tan\theta-h=0$ $\left[\text{Putting } \frac{y}{x}=\tan\theta\right]$.

$\therefore$ $\frac{\tan\theta}{1-\tan^2\theta}=\frac{h}{a-b}$ or $\tan 2\theta=\frac{2b}{a-b}$ which gives the directions of the axes of the conic.

EXAMPLES XX

Find the maximum and minimum values of y given by the following equations :—

1. $y^2+2yx^2+4x-3=0.$ **2.** $y^4+x^4-4xy+2=0.$

3. $y^2-x^2y+x-x^3=0.$

4. Find the point on the parabola $y^2=4ax$ such that its distance from the point $(0, 12a)$ is minimum.

5. Prove that the least value of the portion of the tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ intercepted between the axes, is $a+b$.

6. The greatest rectangle which can be inscribed in a given ellipse has its diagonals along the equiconjugate diameters.

7. The greatest angle at which the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ can be cut by a concentric circle is

$$\tan^{-1}\left|\frac{a^2-b^2}{2ab}\right|$$

8. Show that the shortest normal chord of the parabola $y^2=4ax$ is $6a\sqrt{3}$ and its inclination to the axis is $\tan^{-1}\sqrt{2}$.

9. Find the maximum and minimum values of

$$u=3\sin^2x+4\sin^2y, \text{ where } y-x=\frac{\pi}{4}.$$

10. Find the maximum and minimum values of

(*i*) xy (*ii*) x^2+y^2, where $\frac{x}{a}+\frac{y}{b}=1$.

11. If $\phi(a)=k\neq 0$, and $\phi(a)\neq 0$ and x and y are connected by the relation

$$\phi(x)\phi(y)=k^2$$

prove that the function $f(x)+f(y)$ is a maximum for $x=y=a$ provided that

$$f''(a)<f'(a)\left[\frac{\phi''(a)}{\phi'(a)}-\frac{\phi'(a)}{\phi(a)}\right].$$

12. Two particles P, Q describe fixed straight lines intersecting in O, with constant velocities u and v. Prove that if A, B be simultaneous positions of the particles, and if OA$=a$, OB$=b$, and $\angle$AOB$=w$, the distance PQ will be least after a time

$$\frac{au+bv-(av+bu)\cos w}{u^2-2uv\cos w+v^2}$$

and that the least distance will be

$$\frac{(av-bu)\sin w}{\sqrt{(u^2-2uv\cos w+v^2)}}.$$

13. A ship steaming north at the rate of 12 miles per hour observes a ship, due east of itself and distant 10 miles which is steaming due west at the rate of 16 miles per hour ; after what time are they at the least distance from one another and what is this least distance ?

14. A straight line is drawn through a fixed point (x_1, y_1) meeting the axis OX at A and the axis OY at B, the axes being rectangular and (x_1, y_1) in the positive quadrant ; if the angle OAB be equal to θ, find θ such that (*i*) AB (*ii*) OA+OB (*iii*) OA. OB is a minimum.

15. Assuming that the brightness of a small surface A varies inversely as the square of the distance r from the source of light and directly as the cosine of the angle between r and

the normal to the surface at A, find at what height above the centre of a circle of radius a a source of light should be placed so that the brightness at the circumference should be the greatest.

16. If 40 square feet of sheet metal are to be used in the construction of an open tank with square base, find the dimensions, so that the capacity of the tank is a maximum.

17. An open tank of given volume has a square base. If the inner surface is least possible, find the ratio of the depth to the length.

18. A cone is circumscribed about a sphere of radius r; show that when the volume of the cone is a minimum, its altitude is $4r$ and its semi-vertical angle is $\sin^{-1}\frac{1}{3}$.

19. Show that the altitude of a right circular cone of maximum volume which can be inscribed in a sphere of radius r is $\frac{4r}{3}$. What is the value of the altitude when the area of the curved surface is maximum?

20. Find the volume of the greatest right circular cone that can be described by the revolution about a side of a right-angled triangle of hypotenuse c.

21. Show that the right circular cylinder of the given surface and maximum volume is such that its length is equal to the diameter of the base.

22. A frustum of a cone is l feet long and the radii of its ends are a and b. It is required to cut from it a rectangular block of a square section and of greatest volume. Find the length of the block.

23. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$.

24. A circular cylinder is to be inscribed on a given sphere of radius R. If the total surface of the cylinder, including the two ends, is to be aminimum, show that $h^2=2R^2\left(1-\frac{1}{\sqrt{5}}\right)$

h being the height of the cylinder.

25. Given the sum of the perimeters of a square and a circle, show that the sum of their areas is least when the side of the square is double the radius of the circle.

26. A ladder is to be carried in a horizontal position round a corner formed by two streets a ft. and b ft. wide meeting at right angles. Prove that the length of the longest ladder that will pass round the corner without jamming is $(a^{\frac{2}{3}}+b^{\frac{2}{3}})^{\frac{3}{2}}$ ft.

27. A sector has to be cut from a circular sheet so that the remainder can be formed into a conical shaped vessel of maximum capacity. Find the angle of the sector.

28. Show that the successive maximum values of $y=ae^{-kx}\sin px$ form a series in geometrical progression and that these maxima lie on the curve $y=a\sin \alpha e^{-\frac{kx}{p}}$ where $\tan\alpha=\frac{p}{k}$.

29. If four straight rods be freely hinged at their extremities, show that the quadrilateral of greatest area that they can form is inscribable in a circle.

30. A rectangular sheet of metal has four equal square portions removed at the corners, and the sides are then turned up so as to form an open rectangular box; show that when the volume contained in the box is maximum, the depth will be $\frac{1}{6}\{(a+b)-(a^2-ab+b^2)^{\frac{1}{2}}\}$, where a, b are the sides of the original rectangle.

31. The parcel post regulations restrict parcels to be such that the length plus the girth must not exceed 6 ft., and the length must not exceed $3\frac{1}{2}$ ft. Determine the parcel of greatest volume that can be sent by post, if the form of the parcel be a right circular cylinder. Will the result be affected if the greatest length permitted were only $1\frac{3}{4}$ ft ?

32. Water is poured into a hollow sphere at a uniform rate. Prove that the level of the surface rises at a varying rate, which is least when the surface is level with the centre of the sphere. Show that when the surface bisects a vertical radius, this rate is $\frac{4}{3}$ of what it is at the level of the centre.

[*Hint.* Volume of a spherical cup of depth h formed from a sphere of radius r is given by $V=\frac{\pi}{3}(3rh^2-h^3)$].

CHAPTER VIII

INDETERMINATE FORMS

1. Let $F(x)=f(x)/\phi(x)$, where $f(a)=0$ and $\phi(a)=0$; then $F(a)$ cannot be taken to be equal to $f(a)/\phi(a)$, for this quotient is of the form $0/0$ and is therefore meaningless. The function $F(x)$ takes the indeterminate form $0/0$ for $x=a$. We have already come across such functions. Similarly if $\underset{x\to a}{\text{Lt}}\, f(x)$ is zero and $\underset{x\to a}{\text{Lt}}\, \phi(x)$ is infinite, then the function $F(x)=f(x).\phi(x)$ is said to take the indeterminate form $0\times\infty$ for $x=a$. The other simple indeterminate forms are ∞/∞, $\infty-\infty$, 0^0, 1^∞ and ∞^0.

It should be clearly understood that 0 and ∞ are used here symbolically for the sake of brevity.

It must be specified whether x tends to 'a' in any manner whatsoever or only through values greater than a or only through values less than a. For example, limit of $\frac{\sin x}{x}=1$ when x tends to 0 in any manner whatsoever. Limit of $x\log x=0$ when x tends to 0 but x must tend to zero through positive values only, because the function does not exist for negative values of x.

It may be mentioned that the limit of a function is also sometimes called its **true value**.

Dr. Young is the first to give an excellent account of the theory of Indeterminate Forms and to establish theorems on the forms $0/0$ and ∞/∞ under very general conditions. But his results being beyond the scope of this book cannot be reproduced here. We will therefore content ourselves with proving the rules for the evaluation of the indeterminate forms under very stringent conditions.

2. Form $\frac{0}{0}$.

(A) *Let $f(a)=\phi(a)=0$ and $f'(a)$ and $\phi'(a)$ both exist and be neither both zero nor both infinite. Then the limit (finite or infinite)*

of $\frac{f(x)}{\phi(x)}$ *when* x *tends to* a *is* $\frac{f'(a)}{\phi'(a)}$. *That is to say*

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\frac{f'(a)}{\phi'(a)}$$

For $\operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\operatorname*{Lt}_{h\to 0}\frac{f(a+h)}{\phi(a+h)}$ [putting $x=a+h$, when $x\to a$, $h\to 0$.]

$$=\operatorname*{Lt}_{h\to 0}\frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)}=\operatorname*{Lt}_{h\to 0}\frac{\frac{f(a+h)-f(a)}{h}}{\frac{\phi(a+h)-\phi(a)}{h}}=\frac{f'(a)}{\psi'(a)}.$$

In this rule 'a' is supposed to be finite and it cannot be extended to the case when $a=\infty$. Also if $\phi'(a)=0$, then it follows that the limit is infinite but its sign remains unknown.

(B) *Let* $f(x)$ *and* $\phi(x)$ *be both zero for* $x=a$. *Then*

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\operatorname*{Lt}_{x\to 0}\frac{f'(x)}{\psi'(x)},$$

provided that this limit (finite or infinite) is determinate.

For $\frac{f(a+h)}{\phi(a+h)}=\frac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)}=\frac{f'(a+\theta h)}{\phi'(a+\theta h)}$,

(where θ is a positive proper fraction) by Cauchy's Theorem.

$$\therefore\quad \operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\operatorname*{Lt}_{h\to 0}\frac{f(a+h)}{\phi(a+h)}=\operatorname*{Lt}_{h\to 0}\frac{f'(a+\theta h)}{\phi'(a+\theta h)}$$

$$=\operatorname*{Lt}_{x\to a}\frac{f'(x)}{\phi'(x)}$$

which may be finite or infinite.

It may be observed that if $\frac{f'(a)}{\phi'(a)}$ is still of the form $\frac{0}{0}$ then this rule can be repeated and then we can get

$$\operatorname*{Lt}_{x\to a}\frac{f(x)}{\phi(x)}=\operatorname*{Lt}_{x\to a}\frac{f'(x)}{\phi'(x)}=\operatorname*{Lt}_{x\to a}\frac{f''(x)}{\phi''(x)}$$

Now we shall extend this to the case where $a=+\infty$ and the case where $a=-\infty$. Take for example the case when $a=+\infty$.

$$\underset{x\to+\infty}{\text{Lt}}\frac{f(x)}{\phi(x)}=\underset{y\to+0}{\text{Lt}}\frac{f\left(\frac{1}{y}\right)}{\phi\left(\frac{1}{y}\right)}=\underset{y\to+0}{\text{L}}\frac{f'\left(\frac{1}{y}\right)\left(\times-\frac{1}{y^2}\right)}{\phi'\left(\frac{1}{y}\right)\times\left(-\frac{1}{y^2}\right)}$$

$$=\underset{y\to+0}{\text{Lt}}\frac{f'\left(\frac{1}{y}\right)}{\phi'\left(\frac{1}{y}\right)}=x\to+\infty\ \underset{}{\text{Lt}}\ \frac{f'(x)}{\phi'(x)}.$$

3. We can generalize (A) as follows :—

If $f(x)$ and $\phi(x)$ vanish at $x=a$, and have their first $(n-1)$ derivatives also zero at $x=a$, while their nth derivatives are finite and not both zero, then

$$\underset{x\to a}{\text{Lt}}\frac{f(x)}{\phi(x)}=\frac{f^n(a)}{\phi^n(a)}.$$

Since $f^n(x)$ exists at $x=a$, all the preceding derivatives exist in a close neighbourhood of a.

By the repeated application of Cauchy's Theorem.

$$\underset{h\to0}{\text{Lt}}\frac{f(a+h)}{h^r}=\underset{h\to0}{\text{Lt}}\frac{f(a+h)-f(a)}{h^r-0^r}=\underset{h\to0}{\text{Lt}}\frac{f'(a+h_1)}{rh_1^{r-1}}$$

[where h_1 lies in the interval $(0, h)$]

$$=\ldots\ldots=\underset{h\to0}{\text{Lt}}\frac{f^{r-1}(a+h_{r-1})}{\lfloor r\ h^r_{-1}}$$

[where h_{r-1} lies in the interval $(0, h_{r-2})$]

$$=\frac{1}{\lfloor r}\underset{h\to0}{\text{Lt}}\frac{f^{r-1}(a+h_{r-1})-f^{r-1}(a)}{h_{r-1}}=\frac{1}{\lfloor r}f'(a)$$

as $f^{r-1}(a)=0$.

Similarly $\underset{h\to0}{\text{Lt}}\frac{\phi(a+h)}{h^s}=\frac{1}{\lfloor s}\phi^s(a)$.

But $\frac{f(a+h)}{\phi(a+h)}=h^{r-s}\frac{f(a+h)/h^r}{\phi(a+h)/h^s}$.

Proceeding to the limit, we get

$$\underset{x\to a}{\text{Lt}}\ \frac{f(x)}{\phi(x)}=\frac{f^n(a)}{\phi^n(a)}\ ,\ 0 \text{ or } \pm\infty\ ,$$

according as $r=s=n$, $r>s$, or $r<s$, the sign of infinity depending on the signs of $f^r(a)$ and $\phi^s(a)$.

Ex. 1. Find $\underset{x\to 1}{\text{Lt}}\ \frac{\log x}{x-1}$.

Here $f(x)=\log x$ and $\phi(x)=x-1$.

$\because\ f'(x)=\frac{1}{x}$ and $\phi'(x)=1$,

and $f'(1)=1$ and $\phi'(1)=1$.

$\because\ \frac{f'(1)}{\phi'(1)}=1$ is determinate. Hence by (A)

$$\underset{x\to 1}{\text{Lt}}\ \frac{\log}{x-1}=1.$$

Ex. 2. Find $\underset{x\to 0}{\text{Lt}}\ \frac{1-\cos x}{x^2}$.

Here $f(x)=1-\cos x$, $\phi(x)=x^2$; $f(0)=0$ and $\phi(0)=0$.

$f'(x)=\sin x$, $\phi'(x)=2x$; $f'(0)=0$, $\phi'(0)=0$.

$f''(x)=\cos x$, $\phi''(x)=2$; $f''(0)=1$, $\phi''(0)=2$.

Therefore $\underset{x\to 0}{\text{Lt}}\ \frac{1-\cos x}{x^2}\left(\frac{0}{0}\right)$

$$=\underset{x\to 2}{\text{Lt}}\ \frac{\sin x}{2x}\left(\frac{0}{0}\right)$$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{\cos x}{2}=\frac{1}{2}\ .$$

Ex. 3. Find $\underset{\theta\to 0}{\text{Lt}}\ \frac{e^\theta \sin\theta-\theta-\theta^2}{\theta^3}$

$$\underset{\theta\to 0}{\text{Lt}}\ e^\theta\ \frac{\sin\theta-\theta-\theta^2}{\theta^3}\left(\frac{0}{0}\right)$$

$$= \underset{\theta \to 0}{\text{Lt}} \frac{e^{\theta}(\cos\theta + \sin\theta) - 1 - 2\theta}{3\theta^2} \left(\frac{0}{0}\right)$$

$$= \underset{\theta \to 0}{\text{Lt}} \frac{e^{\theta}(2\cos\theta) - 2}{6\theta} \left(\frac{0}{0}\right)$$

$$= \underset{\theta \to 0}{\text{Lt}} \frac{e^{\theta}(2\cos\theta - 2\sin\theta)}{6} = \frac{1}{3}.$$

EXAMPLES XXI

1. $\underset{x \to 1}{\text{Lt}} \dfrac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}$.

2. $\underset{x \to 0}{\text{Lt}} \dfrac{1 - \sqrt{1 - x^2}}{1 - \cos 2x}$.

3. $\underset{x \to 0}{\text{Lt}} \dfrac{\log \cos x}{x \sin x}$.

4. $\underset{\theta \to \frac{\pi}{4}}{\text{Lt}} \dfrac{\sin\theta - \cos\theta}{\sin 2\theta - \cos 2\theta - 1}$.

5. $\underset{x \to 0}{\text{Lt}} \dfrac{\sin ax}{\sin bx}$.

6. $\underset{x \to 0}{\text{Lt}} \dfrac{a^x - 1}{b^x - 1}$.

7. $\underset{x \to 0}{\text{Lt}} \dfrac{1 - x + \frac{1}{2}x^2 - e^{-x}}{x^3}$.

8. $\underset{x \to 0}{\text{Lt}} \dfrac{a\sin x - \sin ax}{x(\cos x - \cos ax)}$.

9. $\underset{x \to 0}{\text{Lt}} \dfrac{e^x - e^{\sin x}}{x - \sin x}$.

10. $\underset{x \to 0}{\text{Lt}} \dfrac{xe^{2x} - e^{2x} - x + 1}{e^{2x} - 1}$.

11. $\underset{x \to 1}{\text{Lt}} \dfrac{a^{\log x} - x}{\log x}$.

12. $\underset{x \to 0}{\text{Lt}} \dfrac{1 - \cos x}{x \log(1 + x)}$.

13. $\underset{x \to a}{\text{Lt}} \dfrac{e^{nx} - e^{na}}{(x - a)^m}$, m being a positive integer.

14. $\underset{x \to a}{\text{Lt}} \dfrac{\cos x\theta - \cos a\theta}{(x^2 - a^2)^n}$, n being a positive integer greater than 2. Examine the case when (*i*) $n = 1$, (*ii*) $n = 2$.

15. $\underset{x\to\frac{1}{2}}{\text{Lt}} \dfrac{\cos^2\pi x}{e^{2x}-2ex}$.

16. $\underset{\theta\to\frac{\pi}{4}}{\text{Lt}} \dfrac{\sin\left(\frac{\pi}{4}+\theta\right)-1}{\log\sin 2\theta}$.

17. $\underset{x\to 0}{\text{Lt}} \dfrac{e^x+\log\left(\frac{1-x}{e}\right)}{\tan x-x}$.

4. The Form $\left(\frac{\infty}{\infty}\right)$.

(A) Let $f(x)$ and $\phi(x)$ be two functions each of which tends to infinity as x tends to a; suppose further that in the neighbourhood of a, both $f'(x)$ and $\phi'(x)$ exist, are finite and not simultaneously zero and that $\underset{x\to a}{\text{Lt}} \dfrac{f'(x)}{\phi'(x)}$ exists.

Under these conditions, the same rules can be applied to find the limit of $\dfrac{f(x)}{\phi(x)}$ when x tends to a as were used when the indeterminate form was 0/0.

Let c_1 and x, $(c_1<x)$ be two values sufficiently near to a such that $f'(x)$ and $\phi'(x)$ exist and do not simultaneously vanish in the interval (c_1, x). By Cauchy's formula, $\dfrac{f(x)-f(c_1)}{\phi(x)-\phi(c_1)}=\dfrac{f'(c)}{\phi'(c)}$, whence c lies between c_1 and x.

Therefore $$\frac{f(x)}{\phi(x)}\cdot\frac{1-\frac{f(c_1)}{f(x)}}{1-\frac{\phi(c_1)}{\phi(x)}}=\frac{f'(c)}{\phi'(c)}.$$

that is $$\frac{f(x)}{\phi(x)}=\frac{f'(c)}{\phi'(c)}\cdot\frac{1-\frac{\phi(c_1)}{\phi(x)}}{1-\frac{f(c_1)}{f(x)}}.$$

Now if c_1 and x be taken sufficiently near to a, c can be made as near to a as we please then $\dfrac{f'(c)}{\phi'(c)}$ will be as near to its limit l as we please. When this is so, let c_1 be fixed and

x be made to approach a. As $\phi(c_1)$, $f(c_1)$ are fixed and $\phi(x)$, $f(x)$ tend to infinity, therefore $\frac{\phi(c_1)}{\phi(x)}$ and $\frac{f(c_1)}{f(x)}$ tend to zero and

consequently $\dfrac{1-\frac{\phi(c_1)}{\phi(x)}}{1-\frac{f(c_1)}{f(x)}}$ tends to unity.

Hence $$\lim_{x\to a}\frac{f(x)}{\phi(x)}=\lim_{x\to a}\frac{f'(x)}{\phi'(x)}.$$

(B) The previous result can be easily extended to the case when a is infinite. That is, it can be shown that

$$\lim_{x\to\infty}\frac{f(x)}{\phi(x)}=\lim_{x\to\infty}\frac{f'(x)}{\phi'(x)}.$$

Let $x=\frac{1}{y}$.

Therefore $$\lim_{x\to\infty}\frac{f(x)}{\phi(x)}=\lim_{y\to 0}\frac{f\left(\frac{1}{y}\right)}{\phi\left(\frac{1}{y}\right)}$$

$$=\lim_{y\to 0}\frac{f'\left(\frac{1}{y}\right)\left(-\frac{1}{y^2}\right)}{\phi'\left(\frac{1}{y}\right)\left(-\frac{1}{y^2}\right)}=\lim_{y\to 0}\frac{f'\left(\frac{1}{y}\right)}{\phi'\left(\frac{1}{y}\right)}=\lim_{x\to\infty}\frac{f'(x)}{\phi'(x)}.$$

Note. The rule for the evaluation of indeterminate forms $\frac{0}{0}$, $\frac{\infty}{\infty}$, *viz.*, $\lim \frac{f(x)}{\phi(x)} = \lim \frac{f'(x)}{\phi'(x)}$ *should be applied judiciously.*

The following examples illustrate this point :—

(1) $$\lim_{x\to 0}\frac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}=\lim_{x\to 0}\frac{e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)}{e^{\frac{1}{x}}\left(-\frac{1}{x^2}\right)}=1.$$

But the limit of $\dfrac{e^{\frac{1}{x}}-1}{e^{\frac{1}{x}}+1}$ is $+1$ or -1 according as x tends to 0 through positive or negative values.

(2) Let $f(x)=x^2 \sin \dfrac{1}{x}$, $x\neq 0$ and $f(0)=0$; and $\phi(x)=\sin x$.

Now $\underset{x\to 0}{\text{Lt}} \dfrac{f(x)}{\phi(x)}=0$, but applying the above rule

$$\underset{x\to 0}{\text{Lt}} \frac{f'(x)}{\phi'(x)}=\underset{x\to 0}{\text{Lt}} \frac{2x \sin \frac{1}{x}-\cos \frac{1}{x}}{\cos x}=\underset{x\to 0}{\text{Lt}} \left(-\cos \frac{1}{x}\right)$$

which oscillates between -1, to $+1$. *i.e.*, the limit of $\dfrac{f'(x)}{\phi'(x)}$ does not exist.

(3) Let $f(x)=\sin^2 x \sin \dfrac{1}{x}$, $f(0)=0$; $\phi(x)=0$

$$\underset{x\to 0}{\text{Lt}} \frac{f(x)}{\phi(x)}=0.$$

But $\underset{x\to 0}{\text{Lt}} \dfrac{f'(x)}{\phi'(x)}=\underset{x\to 0}{\text{Lt}} \left(\sin 2x \sin \dfrac{1}{x}-\dfrac{\sin^2 x}{x^2} \cos \dfrac{1}{x}\right)$

$=-\underset{x\to 0}{\text{Lt}} \left(\cos \dfrac{1}{x}\right)$ which does not exist.

Ex. 1. Evaluate $\underset{x\to \frac{\pi}{2}}{\text{Lt}} \dfrac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}$.

This is of the form $\dfrac{\infty}{\infty}$. Therefore

$$\underset{x\to \frac{\pi}{2}}{\text{Lt}} \frac{\log\left(x-\frac{\pi}{2}\right)}{\tan x}=\underset{x\to \frac{\pi}{2}}{\text{Lt}} \frac{\frac{1}{x-\frac{\pi}{2}}}{\sec^2 x} \left(\text{still } \frac{\infty}{\infty}\right)$$

$$= \operatorname*{Lt}_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{x - \frac{\pi}{2}} \left(\text{which is of the form } \frac{0}{0} \right)$$

$$= \operatorname*{Lt}_{x \to \frac{\pi}{2}} \frac{-2 \cos x \sin x}{1} = 0.$$

Ex. 2. Evaluate $\operatorname*{Lt}_{\theta \to 0} \dfrac{\log \theta^2}{\log \cot^2 \theta}$.

This is of the form $\dfrac{\infty}{\infty}$, Therefore we have

$$\operatorname*{Lt}_{\theta \to 0} \frac{\log \theta^2}{\log \cot^2 \theta} = \operatorname*{Lt}_{\theta \to 0} \frac{\frac{2}{\theta}}{-\frac{2}{\cot \theta} \operatorname{cosec}^2 \theta}$$

$$= \operatorname*{Lt}_{\theta \to 0} \frac{\sin \theta \cos \theta}{\theta} \left(\frac{0}{0} \text{ form} \right)$$

$$= \operatorname*{Lt}_{\theta \to 0} \frac{\cos 2\theta}{1} = 1.$$

Ex. 3. Evaluate $\operatorname*{Lt}_{x \to \infty} \dfrac{\log x}{x}$.

This is of the form $\dfrac{\infty}{\infty}$.

$$\operatorname*{Lt}_{x \to \infty} \frac{\log x}{x} = \operatorname*{Lt}_{x \to \infty} \frac{1/x}{1} = 0.$$

EXAMPLES XXII

Evaluate the following limits :

1. $\operatorname*{Lt}_{x \to \infty} \dfrac{x^2+1}{x^2+2}$.

2. $\operatorname*{Lt}_{x \to \infty} \dfrac{\log x}{x^m}$, m positive.

3. $\operatorname*{Lt}_{x \to 0} \dfrac{\log \tan ax}{\log \tan x}$.

4. $\operatorname*{Lt}_{x \to 0} \dfrac{\log \sin ax}{\log \sin x}$.

5. $\operatorname*{Lt}_{x \to \infty} \dfrac{x^m}{e^x}$.

6. $\operatorname*{Lt}_{x \to \infty} \dfrac{2^x}{\operatorname{cosec} \frac{a}{2^x}}$.

5. Other Indeterminate Forms. Other Indeterminate forms occasionally arise but, in general, they can always be reduced to the form $\frac{0}{0}$ which, therefore, is a fundamental form; sometimes, however, it may be more convenient to reduce an indeterminate form to the form $\frac{\infty}{\infty}$.

We shall therefore discuss a few other forms very briefly.

Form $0\times\infty$.

(1) If $\underset{x\to a}{\text{Lt}}\, f(x)$ is zero, while $\underset{x\to a}{\text{Lt}}\, \phi(x)$ is infinite, the product $f(x), \phi(x)$ *takes the form* $0\times\infty$; but it can be made to take the form $\frac{0}{0}$ by writing it as

$$f(x),\phi(x)=\frac{f(x)}{1/\phi(x)}.$$

Form $\infty-\infty$.

(2) If $\underset{x\to 0}{\text{Lt}}\, f(x), \underset{x\to a}{\text{Lt}}\, \phi(x)$ are each infinite, then the function $f(x)-\phi(x)$ *takes the form* $\infty-\infty$; but it can be made to take the form $0/0$ by writing it as

$$f(x)-\phi(x)=\left(\frac{1}{\phi(x)}-\frac{1}{f(x)}\right)\Big/\frac{1}{f(x)\phi(x)}.$$

Forms ∞°, 0° and 1^∞.

(3) If $\underset{x\to a}{\text{Lt}}\, f(x)$ is zero, while $\underset{x\to a}{\text{Lt}}\, \phi(x)$ is infinite and we are required to find $\underset{x\to a}{\text{Lt}}\,[\phi(x)]^{f(x)}$, which *takes the form* ∞°, we proceed as follows :

If $\phi(x)$ is positive let $y=[\phi(x)]^{f(x)}$
then $\log y=f(x)\log\,\phi(x)$

$$=\frac{f(x)}{\frac{1}{\log\,\phi(x)}},$$ which is of the form $\frac{0}{0}$

or $$= \frac{\log \phi(x)}{\frac{1}{f(x)}}, \text{ which is of the form } \frac{\infty}{\infty}.$$

If $\log y$ has a limit, say l, then since e^x is a continuous function of x,

$$\underset{x\to a}{\text{Lt}}\ y = \underset{x\to a}{\text{Lt}} [\phi(x)^{f(x)}] = e^l.$$

The Forms 0°, 1^∞ can be treated in the same manner.

Ex. 1. Evaluate $\underset{x\to +0}{\text{Lt}}\ x^x (0^\circ)$. .

Let $x^x = y$. Therefore $x \log x = \log y$.

$$\underset{x\to +0}{\text{Lt}}\ \log y = \underset{x\to +0}{\text{Lt}}\ x \log x\ (0, \infty)$$

$$= \underset{x\to +0}{\text{Lt}} \frac{\log x}{\frac{1}{x}} \left(\frac{\infty}{\infty}\right) = \underset{x\to +0}{\text{Lt}} \frac{x}{-\frac{1}{x^2}} = \underset{x\to +0}{\text{Lt}}\ -x = 0$$

$\therefore\ y = 1$. Hence $\underset{x\to +0}{\text{Lt}}\ x^x = 1$.

Ex. 2. Evaluate $\underset{x\to \frac{\pi}{2}}{\text{Lt}}\ (\tan x)^{\sin 2x}\ (\infty^\circ)$

Let $(\tan x)^{\sin 2x} = y$. Then

$\log y = \sin 2x \log (\tan x)$.

$$\underset{x\to \frac{\pi}{2}}{\text{Lt}}\ \log y = \underset{x\to \frac{\pi}{2}}{\text{Lt}}\ \sin 2x \log \tan x\ (0, \infty)$$

$$= \underset{x\to \frac{\pi}{2}}{\text{Lt}} \frac{\log \tan x}{\operatorname{cosec} 2x} \left(\frac{\infty}{\infty}\right)$$

$$= \underset{x\to \frac{\pi}{2}}{\text{Lt}} \frac{\frac{\sec^2 x}{\tan x}}{-2 \operatorname{cosec} 2x \cot 2x}$$

$$= \underset{x\to\frac{\pi}{2}}{\text{Lt}} \ (-\tan 2x) = 0$$

$\therefore \quad y = 1.$

EXAMPLES XXIII

Evaluate :

1. $\underset{x\to 0}{\text{Lt}} \left(\frac{1}{x^2} - \cot^2 x \right)$.

2. $\underset{x\to 0}{\text{Lt}} \left(\frac{1}{x^2} - \frac{\cot x}{x} \right)$.

3. $\underset{x\to 0}{\text{Lt}} \ (\cos bx)^{\operatorname{cosec}^2 ax}$.

4. $\underset{x\to\infty}{\text{Lt}} \left(\frac{\pi}{2} - \tan^{-1} x \right)^{\frac{1}{x}}$.

5. $\underset{\theta\to\frac{\pi}{4}}{\text{Lt}} \ (\tan \theta)^{\tan 2\theta}$.

6. $\underset{x\to 0}{\text{Lt}} \left(\frac{\tan x}{x} \right)^{\frac{1}{x^2}}$.

7. $\underset{x\to 0}{\text{Lt}} \left(a^x + x \right)^{\frac{1}{x}}$

8. $\underset{x\to 0}{\text{Lt}} \left(\cos x \right)^{\frac{1}{x}}$; $\left(\cos x \right)^{\frac{1}{x^2}}$; $\left(\cos x \right)^{\frac{1}{x^3}}$

9. Show that (*i*) $\underset{t\to\infty}{\text{Lt}} (1+t)^{\frac{1}{t}} = 1$. (*ii*) $\underset{x\to\infty}{\text{Lt}} (1-x)^{-\frac{1}{x}} = 1$.

10. Show that $\underset{x\to\infty}{\text{Lt}} \left(1 + \frac{a}{x} \right)^x = e^a$.

6. Utilisation of Taylor's Theorem. In those cases where the indeterminate character of a function does not appear by the repeated application of the previous form, a finite judicious application of Taylor's Theorem gives the result more rapidly than the rules referred to, but we should have recourse to other methods when the application of Taylor's Theorem to a finite form is not convenient.

$f(x)$ a_2. $t)\}$ $f(x)$ [illegible] ames the

[illegible]eterminate.

For example, let it be required to find $\underset{x\to 0}{\text{Lt}} \dfrac{x-\sin x}{x^3}$

$$\underset{x\to 0}{\text{Lt}} \frac{x-\sin x}{x^3}$$

$$= \underset{x\to 0}{\text{Lt}} \frac{x-\left\{x-\dfrac{x^3}{|3}+\dfrac{x^3}{|5}\cos\theta x\right\}}{x^3}$$

$$= \underset{x\to 0}{\text{Lt}} \left(\frac{1}{|3}-\frac{x^2}{|5}\cos\theta x\right) = \frac{1}{6}.$$

EXAMPLES XXIV

1. Show that $\underset{x\to\infty}{\text{Lt}} \dfrac{x^2}{x-\sin x}$ cannot be evaluated by the rules even though the limit is infinite.

Evaluate

2. $\underset{x\to 0}{\text{Lt}} \dfrac{\log\tan ax}{\log\tan bx}$.

3. $\underset{x\to 0}{\text{Lt}} \dfrac{\log\tan ax-\log\tan bx}{\log\sin ax-\log\sin bx}$.

4. $\underset{x\to 0}{\text{Lt}} \dfrac{\sinh x-\sin x}{x^3}$.

5. $\underset{x\to 0}{\text{Lt}} \dfrac{\cosh x-\cos x}{x^2}$.

6. $\underset{x\to a}{\text{Lt}} \dfrac{(x-a)^{\frac{1}{m}}}{(x^2-a^2)^{\frac{1}{n}}}$, m and n being positive integers, $m<n$.

7. $\underset{x\to 0}{\text{Lt}} \dfrac{\tan\pi x-\pi x}{2x^2\tan\pi x}$.

8. $\underset{x\to 0}{\text{Lt}} \dfrac{\tan x-\sin x}{\sin^3 x}$.

9. $\underset{x\to 0}{\text{Lt}} (\cos x)^{\cot^2 x}$.

10. $\underset{x\to \frac{\pi}{2}}{\text{Lt}} \left(\frac{\pi}{2}-x\right)^{\tan x}$.

11. $\underset{x\to 1}{\text{Lt}} \left(1-\dfrac{2x}{\pi}\right)^{\tan x}$.

12. $\underset{x\to 1}{\text{Lt}} \sec\dfrac{\pi x}{2}\log\dfrac{1}{x}$.

13. $\underset{x\to 2}{\text{Lt}} 2^x \sin\dfrac{a}{2^x-2}$.

14. $\underset{x\to 1}{\text{Lt}} \left(\dfrac{x}{x-1}-\dfrac{1}{\log x}\right)$.

15. $\underset{x\to 0}{\text{Lt}} \quad \dfrac{\pi x-1}{2x^2}+\dfrac{\pi}{x(e^{2\pi x}-1)}$.

16. $\underset{x\to 0}{\text{Lt}} \left(\dfrac{1}{x^m}\right)^{x^n}$

17. $\underset{\theta\to \frac{\pi}{2}}{\text{Lt}} \cos\theta^{\cos\theta}$.

18. $\underset{x\to 0}{\text{Lt}} \quad \dfrac{3\tan x-3x-x^3}{x^5}$.

19. $\underset{x\to 0}{\text{Lt}} \quad \dfrac{(1+x)^{\frac{1}{x}}-e}{x}$.

20. Show that $\underset{x\to 0}{\text{Lt}} \quad \dfrac{(1+x)^{\frac{1}{x}}-e+\frac{ex}{2}}{x^2}=\dfrac{11e}{24}$.

21. $\underset{x\to 0}{\text{Lt}} \quad \dfrac{\sin 2x+2\sin^2 x-2\sin x}{\cos x-\cos^2 x}$.

22. $\underset{x\to\infty}{\text{Lt}} \; x\tan\dfrac{1}{x}$.

23. $\underset{x\to 0}{\text{Lt}} \quad \dfrac{xe^x-\log(1+x)}{x^2}$.

24. $\underset{m\to n}{\text{Lt}} \quad \dfrac{\tan mx-\tan nx}{\sin^2 mx-\sin^2 nx}$.

25. $\dfrac{\sin^2 mx-\sin^2 nx}{1-\cos(m-n)x}$, when (*i*) $n\to 0$ (*ii*) $m\to n$.

26. $\dfrac{2f(x+h)-2f(x)-2hf'(x)-h^2f'(x)}{h\{f(x+h)-f(x)-hf'(x)\}}$ when $h\to 0$.

27. $\underset{x\to\infty}{\text{Lt}} \quad \dfrac{Ax^n+Bx^{n-1}+Cx^{n-2}+\ldots\ldots}{ax^m+bx^{m-1}+cx^{m-2}+\ldots\ldots}$ according as n is $>$, $=$, or $<m$.

28. If $\underset{x\to 0}{\text{Lt}} \quad \dfrac{\sin 2x+a\sin x}{x^3}$ be finite, find the value of a and the limit.

29. If $\underset{x\to 0}{\text{Lt}} \quad \dfrac{\sinh 3x+a_1\sinh 2x+a_2\sinh x}{x^3}$ have a finite limit, find its and determine the necessary values of a_1 and a_2.

30. Show that the expression $\{\phi(x)\}^{f(x)}$ which assumes the form 0^∞ when $x\to a$ is not indeterminate.

CHAPTER VIII

TANGENTS AND NORMALS TO PLANE CURVES

1. Tangent at a point. *The tangent to a plane curve at the point P has already been defined to be the limit (if it exists) of the straight line PQ when Q tends to P along the curve.*

Unless P is a singular point on the curve, the position of tangent PT is the same whether Q approaches P from the right or from the left.

Equation of the tangent. *Assuming the existence of the tangent to the curve $y=f(x)$ at the point $P(x, y)$, its equation can be easily found.*

Let $Q(x+\delta x, y+\delta y)$ be a neighbouring point on the curve. Then the equation to the secant PQ is

$$\frac{Y-y}{X-x}=\frac{(y+\delta y)-y}{(x+\delta x)-x}=\frac{\delta y}{\delta x},$$

where X, Y are current co-ordinates of an arbitrary point on the secant. Now let Q tend to P. Then $\delta x \to 0$ and the straight line PQ tends to the straight line

$$\text{Lt}\,\frac{Y-y}{X-x}=\underset{\delta x\to 0}{\text{Lt}}\,\frac{\delta y}{\delta x}, \qquad \ldots\ldots(1)$$

since the arbitrary point X, Y on the secant becomes an arbitrary point on the tangent, which may still be denoted by X, Y when $\delta x \to 0$.

Hence from (1) the equation of the tangent at the point (x, y) is

$$\frac{Y-y}{X-x}=\frac{dy}{dx},$$

or $\mathbf{Y-y=f'(x)(X-x)}$.

This proof is applicable whether the axes are rectangular or oblique.

2. When the equation of the curve is given in the form $f(x, y)=0$, $\frac{dy}{dx}=-\frac{\partial f}{\partial x}\Big/\frac{\partial f}{\partial y}$ and the equation of the tangent at the point (x, y) after substituting this value of $\frac{dy}{dx}$ becomes,

$$(X-x)\frac{\partial f}{dx}+(Y-y)\frac{\partial f}{\partial y}=0 \qquad \ldots\ldots(2)$$

It is assumed here that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are not both zero.

3. Equation of the normal (Rectangular Axes).

The normal to a plane curve at the point P is a straight line through P at right angles to the tangent to the curve at P.

Slope of the tangent at P(x, y) is $\frac{dy}{dx}$.

$\therefore$ Slope of normal is $-1\Big/\frac{dy}{dx}$ (*i. e.*, minus the reciprocal of the slope of the tangent).

Hence the general equation of the normal at (x, y) is

$$(X-x)+\frac{dy}{dx}(Y-y)=0.$$

$\therefore$ for the curve $y=f(x)$, the equation of the normal at (x, y) is

$$\frac{X-x}{\frac{\partial f}{\partial x}}=\frac{Y-y}{\frac{\partial f}{\partial y}}.$$

Ex. 1, Find the equations of the tangents and the normals to the ellipse $9x^2+y^2=25$ at the points $(1, 4)$ and (x', y').

Differentiating $9x^2+y^2-25=0$ with respect to x, we get $\frac{dy}{dx}=-9\frac{x}{y}$.

$$\therefore \quad \left[\frac{dy}{dx}\right]_{\substack{x=1\\y=4}}=\left[-9\frac{x}{y}\right]_{\substack{x=1\\y=4}}=-\frac{9}{4}.$$

and $\left(\frac{dy}{dx}\right)_{\substack{x=x'\\y=y'}} = -9\,\frac{x'}{y'}$.

The equations of the tangents are, therefore

$$y-4=-\frac{9}{4}(x-1) \text{ or } 9x+4y-25=0 \text{ at } (1, 4)$$

and $y-y'=-9\,\frac{x'}{y'}(x-x')$ or $9xx'+yy'=9x'^2+y'^2=25$,

because the point (x', y') satisfies the equation

$$9x^2+y^2=25.$$

Again, the equations of the normals are

$y-4=\frac{4}{9}(x-1)$ or $4x-9y+32=0$ at $(1, 4)$

and $x-y'=\frac{y'}{9x'}(x-x')$ or $y'x-9x'y+8x'y'=0$ at (x', y').

Ex. 2. Find the equations of the tangent and the normal at (x, y) of the conic.

$$f(x, y)\equiv ax^2+2hxy+by^2+2gx+2fy+c=0.$$

The equation of the tangent is

$$(X-x)\frac{\partial f}{\partial x}-(Y-y)\frac{\partial f}{\partial y}=0.$$

i. e., $X\frac{\partial f}{\partial x}-Y\frac{\partial f}{\partial y}-\left(x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right)=0.$

Now $\frac{\partial f}{\partial x}=2(ax+hy+g)$ and $\frac{\partial f}{\partial y}=2(hx+by+f)$.

Hence the equation of the tangent becomes

$$(ax+hy+g)X+(hx+by+f)Y - \{x(ax+hy+g)+y(hx+by+f)\}=0.$$

$$(ax+hy+g)X+(hx+by+f)Y - \{ax^2+2hxy+by^2+gx+fy\}=0 \quad \ldots(A)$$

But from the equation of the curve

$$ax^2+2hxy+by^2+gx+fy=-gx-fy-c.$$

$\therefore$ (A) becomes

$$(ax+hy+g)X+(hx+by+f)Y+gx+fy+c=0,$$

which is the equation of the tangent.

The equation of the normal is

$$\frac{X-x}{ax+hy+g}=\frac{Y-y}{hx+by+f}.$$

Ex. 3. Find the equation of the tangent at (x, y) to the curve $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$.

The equation of the tangent is

$$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0,$$

where $f(x, y)=\frac{x^m}{a^m}+\frac{y^y}{b^m}-1=0,$

or $(X-x)\frac{mx^{m-1}}{a^m}+(Y-y)\frac{my^{m-1}}{b^m}=0,$

or $X\frac{x^{m-1}}{a^m}+Y\frac{y^{m-1}}{b^m}-\left(\frac{x^m}{a^m}+\frac{y^m}{b^m}\right)=0.$

Hence the tangent is

$$\frac{x^{m-1}}{a^m}X+\frac{y^{m-1}}{b^m}Y=1.$$

Ex. 4. Find the condition that the straight line $x\cos\alpha+y\sin\alpha=p$ may touch the curve $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$.

By the preceding example, the equation of the tangent at (x_1,y_1) is $\frac{x_1^{m-1}}{a^m}x+\frac{y_1^{m-1}}{b^m}y=1.$

Comparing this with the equation $x\cos\alpha+y\sin\alpha=p$ we get

$$\frac{\cos\alpha}{p}=\frac{x_1^{m-1}}{a^m}\text{ and }\frac{\sin\alpha}{p}=\frac{y_1^{m-1}}{b^m}.$$

i. e., $\frac{a\cos\alpha}{p}=\left(\frac{x_1}{a}\right)^{m-1}$ and $\frac{b\sin\alpha}{p}=\left(\frac{y_1}{b}\right)^{m-1}$

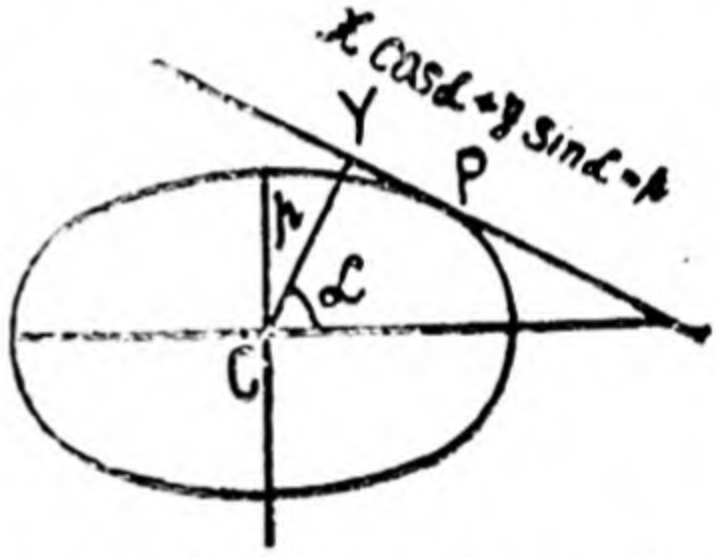
$$\therefore \left(\frac{a\cos\alpha}{p}\right)^{\frac{1}{m-1}} = \frac{x_1}{a} \text{ and } \left(\frac{b\sin\alpha}{p}\right)^{\frac{1}{m-1}} = \frac{y_1}{b}.$$

Hence substituting in the equation of the curve these values of x_1/a and y_1/b, we get

$$\left(\frac{a\cos\alpha}{p}\right)^{\frac{m}{m-1}} + \left(\frac{b\sin\alpha}{p}\right)^{\frac{m}{m-1}} = 1.$$

$$i.\ e.,\ (p)^{\frac{m}{m-1}} = (a\cos\alpha)^{\frac{m}{m-1}} + (b\sin\alpha)^{\frac{m}{m-1}}. \qquad (A)$$

Cor. 1. Putting $m=2$, we get $p^2=a^2\cos^2\alpha+b^2\sin^2\alpha$ as the condition that the line $x\cos\alpha+y\sin\alpha=p$ may touch the ellipse $x^2/a^2+y^2/b^2=1$.



Cor. 2. It will be observed that the r co-ordinate of the foot of the perpendicular is p and the θ-co-ordinate is α. Therefore, the condition (A) expresses a relation between the polar co-ordinates of the foot of the perpendicular. Hence the condition (A) itself is the equation of the locus of the foot of the perpendicular in polar co-ordinates. Writing in the customary form, that is, changing p into r and α into θ, we get the *polar equation of the locus of the foot of the perpendicular from the origin on the tangent as*

$$r^{\frac{m}{m-1}} = (a\cos\theta)^{\frac{m}{m-1}} + (b\sin\theta)^{\frac{m}{m-1}}.$$

By putting $\cos\theta=x/r$ and $\sin\theta=y/r$, the Cartesian equation on the locus is

$$(x^2+y^2)^{\frac{m}{m-1}} = (ax)^{\frac{m}{m-1}} + (by)^{\frac{m}{m-1}}.$$

4*. Oblique axes. Find the equation of the normal at (x, y) to the curve $f(x, y) = 0$ when the axes are oblique.

If ψ be the angle the tangent to the curve at (x, y) makes with the positive direction of the x-axis, then

$\frac{dy}{dx} = \tan\psi$ in rectangular co-ordinates. But in oblique axes

$\frac{dy}{dx}$ is not equal to $\tan\psi$.

$\frac{dy}{dx} = \frac{\sin\psi}{\sin(\omega-\psi)}$ in oblique axes, where ω is the angle between the axes,

$$\text{whence } \tan\psi = \frac{\frac{dy}{dx}\sin\omega}{1+\frac{dy}{dx}\cos\omega}.$$

We know, from Co-ordinate Geometry, that if $y=mx+c$ and $y=m'x+c'$ represent two straight lines where axes are oblique, then the angle θ between these lines is given by

$$\tan\theta = \frac{(m-m')\sin\omega}{1+(m+m')\cos\omega+mm'}$$

and the condition that the two lines should be at right angles is that $1+(m+m')\cos\omega+mm'=0$;

$$\text{whence } m' = -\frac{1+m\cos\omega}{\cos\omega+m} \text{ in terms of } m.$$

The equation of tangent is $Y-y=\frac{dy}{dx}(X-x)$ whether the axes are rectangular or oblique ; thus m of the tangent is $\frac{dy}{dx}$.

$$\therefore \quad m \text{ of the normal} = m' = -\frac{1+\frac{dy}{dx}\cos\omega}{\cos\omega+\frac{dy}{dx}}.$$

Hence the equation of the normal (x, y) is

$$Y-y = -\frac{1+\cos\omega\frac{dy}{dx}}{\cos\omega+\frac{dy}{dx}}(X-x) \text{ in oblique axes.}$$

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5. Simplification for Algebraic curves. In the equation of the tangent

$$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}=0.$$

the absolute term $=-x\frac{\partial f}{\partial x}-y\frac{\partial f}{\partial y}$.

This can be written in a simplified form by adopting the following artifice :

Let the equation of the curve $f(x, y)=0$ be made homogeneous by the introduction of suitable powers of z so that the function $f(x, y)$ of degree n becomes a homogeneous function $f(x, y, z)$.

Then, by Euler's Theorem,

$$x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+ +z\frac{\partial f}{\partial z}=nf(x, y, z)=0.$$

$$\therefore -x\frac{\partial f}{\partial x}-y\frac{\partial f}{dy}=z\frac{\partial f}{\partial z}$$

Thus the equation of the tangent takes the form

$$X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}=0,$$

where z is to be put equal to 1, after differentiation has been performed ; or writing Z for z in the last term, for the sake of symmetry, the equation of the tangent takes the more symmetrical form

$$X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}=0,$$

where in the last term both Z and z are to be equal to 1 after differentiation has been performed. It is easier to remember the equation of the tangent in this form which has the further advantage that the absolute term comes out automatically in its simplified form.

Ex. Find the equation of the tangent at (x, y) to the conic $f(x, y)=ax^2+2hxy+by^2+2gx+2fy+c=0$.

Making the equation homogeneous by the introduction of x, we have

$$f(x, y, z)=ax^2+2hxy+by^2+(2gx+2fy)z+cz^2.$$

$$\therefore \frac{\partial f}{\partial x}=(ax+hy+gz).$$

$$\frac{\partial f}{\partial y}=2(hx+by+fz),\ \frac{\partial f}{\partial z}=2(gx+fy+cz).$$

Hence the equation of the tangent is

$$X\frac{\partial f}{\partial x}+Y\frac{\partial f}{\partial y}+Z\frac{\partial f}{\partial z}=0,$$

i. e., $2(ax+hy+gz)X+2(hx+by+fz)Y+2(gx+fy+cz)Z=0.$

Hence putting $Z=1$ and $z=1$, we get

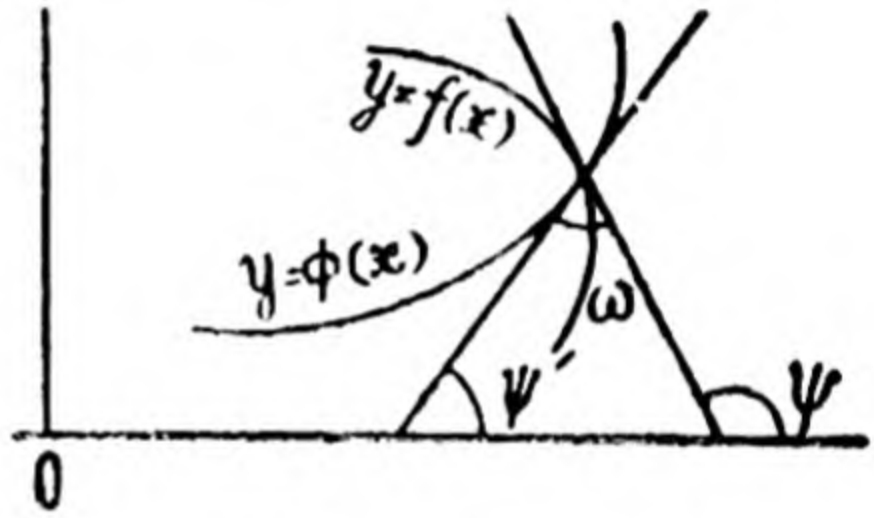
$$(ax+hy+g)X+(hx+by+f)Y+(gx+fy+c)=0$$

as the equation of the tangent at $(x, y.)$

6. Angle of intersection of two curves.

The angle at which two curves intersect is the angle between their tangents at the point of intersection.

Let $y=f(x)$ and $y=\phi(x)$ be two curves. We find the co-ordinates of the points of intersection by solving for x and y from these equations.



If the curves intersect at an angle ω, then $\omega=\psi-\psi'$

$$\therefore\quad \tan\omega=\frac{\tan\psi-\tan\psi'}{1+\tan\psi\tan\psi'}=\frac{f'(x)-\phi'(x)}{1+f'(x)\phi'(x)}.$$

Cor. The two curves *cut orthogonally* (*at right angles*) if $f'(x),\phi f'(x)=-1$ and touch if $f'(x)=\phi'(x)$.

Ex. 1. Find the angle of intersection of the parabola $y^3=2x$ and the circle $x^2+y^2=8$.

Solving simultaneously, we find two real points of intersection of the two curves namely, (2, 2) and (2, −2).

(*i*) At (2, 2), the values of $\frac{dy}{dx}$ obtained from the equations of the two curves are

$$\left|\frac{1}{(2x)^{\frac{1}{2}}}\right|_{x=2} \quad \text{and} \quad \left|-\frac{x}{y}\right|_{\substack{x=2\\y=2}}$$

i.e., $\frac{1}{2}$ and -1 are the slopes of the two tangents at (2, 2) belonging to the two curves.

Since the angle between two lines whose slopes are m_1 and m_2 is given by the formula $\tan\theta=\dfrac{m_1-m_2}{1+m_1m_2}$,

$\therefore$ the acute angle between the two tangents is $\tan^{-1}\dfrac{\frac{1}{2}-(-1)}{1+(-1).\frac{1}{2}}=\tan^{-1}3$. This is also the angle of intersection at (2, -2) as is evident from considerations of symmetry.

Ex. 2. If the conics $a_1x^2+b_1y^2=1$ and $a_2x^2+b_2y^2=1$ intersect at right angles, show that

$$\frac{1}{a_1}-\frac{1}{b_1}=\frac{1}{a_2}-\frac{1}{b_2}.$$

Let (x', y') be one of the points of intersection of the two conics.

$\dfrac{dy}{dx}=-\dfrac{a_1x}{b_1y}$ from the first conic.

(*i*) $\therefore \left|\dfrac{dy}{dx}\right|_{(x',\ y')}=-\dfrac{a_1x'}{b_1y'}$ [slope of tangent at (x', y') to $a_1x^2+b_1y^2=1$].

Similarly, (*ii*) $\left|\dfrac{dy}{dx}\right|_{(x',\ y')}=-\dfrac{a_2x'}{b_2y'}$, [slope of tangent at (x', y') to $a_2x^2+b_2y^2=1$.]

Since the tangents cut at right angles,

$$-\frac{a_1x'}{b_1y'}\times-\frac{a_2x'}{b_2y'}=-1\ (\because\ m_1\ m_2)=-1)$$

i.e., $a_1\,a_2\,x'^2+b_1\,b_2\,y'^2=0.$ (1)

But since $(x'\ y')$ is a common point of intersection of the conics

$$a_1\,x'^2+b_1\,y'^2-1=0 \quad (2)$$

and

$$a_1\,x'^2+b_2\,y'^2-1=0. \quad (3)$$

x'^2 and y'^2 are to be eliminated between (1), (2) and (3). From (2) and (3) by cross-multiplication

$$\frac{x'^2}{-b_1+b_2}=\frac{y'^2}{-a_2+a_1}=\frac{1}{a_1b_2-a_2b_1}$$

$$\therefore\quad x'^2=\frac{b_2-b_1}{a_1b_2-a_2b_1},\quad y'^2=\frac{a_1-a_2}{a_1b_2-a_2b_1}.$$

Substituting for x'^2 and y'^2 in (1), we get

$$\frac{a_1a_2(b_2-b_1)+b_1b_2(a_1-a_2)}{a_1b_2-a_2b_1}=0.$$

Making the numerator zero, we get

$$\frac{a_1-a_2}{a_1a_2}=\frac{b_1-b_2}{b_1b_2}\quad\text{or}\quad\frac{1}{a_2}-\frac{1}{a_1}=\frac{1}{b_2}-\frac{1}{b_1}.$$

$$i.e.,\quad\frac{1}{a_1}-\frac{1}{b_1}=\frac{1}{a_2}-\frac{1}{b_2}.$$

7. Parametric equations of a curve. If a curve be given by its parametric equations (also called its freedom equations)

$$\left.\begin{aligned}x&=\phi(t)\\ y&=\psi(t)\end{aligned}\right]$$

we have $\dfrac{dy}{dx}=\dfrac{dy}{dt}\Big/\dfrac{dx}{dt}=\dfrac{\psi'(t)}{\phi'(t)}$.

Hence the *equations of the tangent and the normal* at t are

$$\phi'(t)\{y-\psi'(t)\}-\psi'(t)\{x-\phi(t\}=0$$

and $$\phi'(t)\{x-\phi(t)\}+\psi(t)\{y-\psi(t)\}=0.$$

It is assumed that $\phi'(t)$ and $\psi'(t)$ are not simultaneously zero at t.

Ex. Find the equations of the tangent and the normal to the ellipse $x=a\cos\phi$, $y=b\sin\phi$ at the point ϕ.

Here ϕ is the parameter and the Cartesian equation of the ellipse is obtained by eliminating ϕ between the two equations. Thus,

$\left(\dfrac{x}{a}\right)^2+\left(\dfrac{y}{b}\right)^2=\cos^2\phi+\sin^2\phi=1$ which is the cartesian

equation of the ellipse.

We have $x=a\cos\phi \therefore \frac{dx}{d\phi}=-a\sin\phi$

$y=b\sin\phi \therefore \frac{dy}{d\phi}=b\cos\phi.$

$\therefore \frac{dy}{dx}=\frac{dy}{d\phi}\Big/\frac{dx}{d\phi}=-\frac{b}{a}\cot\phi.$

Hence the equation of the tangent is

$$y-b\sin\phi=-\frac{b}{a}\cot\phi(x-a\cos\phi),$$

or $a\sin\phi.y+b\cos\phi.x=ab\ (\sin^2\phi+\cos^2\phi).$

Thus $\frac{x\cos\phi}{a}+\frac{y\sin\phi}{b}=1$ is the tangent at ϕ. And the equation of the normal at ϕ is

$$y-b\sin\phi=\frac{a}{b}\tan\phi(x-a\cos\phi)$$

or $$\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}=a^2-b^2.$$

EXAMPLES XXV

1. Find the equations of the tangent and the normal at any point on each of the following curves : —

(*i*) $x^3y+xy^2=b^3$. (*ii*) $xy=k^2$. (*iii*) $y=a\cos h\frac{x}{a}$.

(*iv*) $(x^2+y^2)^2=a^2(x^2-y^2)$. (*v*) $x^5+y^5=a^5$.

2. Find the tangent and the normal at any point on the following curves :—

(*i*) $x=a\cos\theta$. $y=b\sin\theta$.

(*ii*) $x=at^2$. $y=2at$.

(*iii*) $x=at(\theta+\sin\theta)$ $y=a(1-\cos\theta)$.

(*iv*) $x=a\cos\theta+a\theta\sin\theta$, $y=a\sin\theta-a\theta\cos\theta$.

(*v*) $x=3\cos\theta-\cos 3\theta$, $y=3\sin\theta-\sin 3\theta$.

3. Prove that the line

$$\frac{x}{a}+\frac{y}{b}=1 \text{ touches the curve } y=be^{-\frac{x}{a}}$$

at the point where the curve crosses the y-axis.

4. If a normal to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ makes an angle θ with the x-axis, show that its equation is

$$y\cos\theta-x\sin\theta=a\cos 2\theta.$$

5. Find the points on the curve

$$y=(x-1)(x-2)(x-3)$$

at which the tangents are parallel to the axis of x.

6. The curve $y=x(x-a)^2$ is cut in three points by the line $y=m^2x$. Find the tangents at those points and determine the points where they cut the curve again.

7. Prove that the tangents to curve $x^3+y^3=3axy$ at the two points in which it is cut by the straight lines $y=\pm mx$ intersect on the curve.

8. Determine the abscissae of the points at which the tangents to the curve $y=x^3$ are parallel to the line $y=3x+5$.

9. Find the equation of the tangent and the normal at any point P of the curve $x^3+y^3=c^3$.

Prove that the area of a triangle formed by the normal and the axes is equal to the square on the perpendicular from P to one of the bisectors of the angle between the axes.

10. Show that the equation of the tangent at (h, k) to the algebraic curve $f(x, y)=0$ is $xf_h(h, k)+y\ f_k(h, k)+\phi\ (h, k)=0$, where $\phi(h, k)$ is formed from $f(h, k)$ by multiplying each term of the latter by the number by which its degree falls short of that of the highest term.

Write down the equation of the tangent to the curve

$$x^3-3xy+y-1=0.$$

11. Show that if $x\cos\alpha+y\sin\alpha=p$ touches the curve

$$x^m y^n=a^{m+n}$$

then $p^{m+n}\ m^m\ n^n-(m+n)^{m+n}a^{m+n}\cos^m\alpha\sin^n\alpha$.

If $Ax+By=C$ is a normal to the curve $a^{n-1}\ y=x^n$, show that

$$n^n\ A^n\ C^{n-1}=a^{n-1}B(nA^2+B^2)^{n-1}.$$

12. If $p=x\cos\alpha+y\sin\alpha$ touches the curve

$$\left(\frac{x}{a}\right)^{\frac{n}{n-1}}+\left(\frac{y}{b}\right)^{\frac{n}{n-1}}=1,$$

prove that $p^n=(a\cos\alpha)^n+(b\sin\alpha)^n$.

13. Find the condition that the line $y=mx+c$ may be a normal to the curve $x^4+y^4=a^4$.

14. Show that the parabolas $x^2=ay$ and $y^2=2ax$ intersect on the curve $x^3+y^3=3axy$; also find the angles between each pair at the points of intersection.

15. Show that the curves $x^3-3xy^2=-2$ and $3x^2y-y^3=2$ cut orthogonally.

16. Show that the curves $y^2=4ax$ and $ay^2=4x^3$ intersect each other at an angle $\tan^{-1}1/2$; also if PG_1 and PG_2 be the normals to the two curves at a common point of intersection (other than the origin) meeting the axis of x in G_1 and G_2, then $G_1G_2=4a$.

17. Find the angle of intersection of the curves $x^2-y^2=r^2$, $x^2+y^2=\sqrt{2}r^2$.

18. Show that the curves

$$\frac{x^2}{a^2+\lambda_1}+\frac{y^2}{b^2+\lambda_1}=1 \text{ and } \frac{x^2}{a^2+\lambda_2}+\frac{y^2}{b^2+\lambda_2}=1$$

intersect at right angles.

19. Show that there exist points P and Q on the curve $y^2=x^3$ such that the line PQ is both tangent and normal to the curve.

20. Show that if $f(x)$ is continuous at x_0, the tangent at $P(x_0, y_0)$ to the curve $y=f'(x)$ is the limit of the chord Q_1Q_2 when Q_1, Q_2 tend to P along the curve.

[*Hint.* The equation of the chord Q_1Q_2 is

$$y-f(x_1)=\frac{f(x_2)-f(x_1)}{x_2-x_1}(x-x_1).$$

Since $f'(x)$ is continuous at x_0 therefore it is implied that it exists at all neighbouring points to P. Hence Q_1 and Q_2 may be taken near enough to P for $f'(x)$ to exist at all points of the

interval $x_1<x<x_2$. Therefore by the mean value theorem

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'\{x_1+\theta(x_2-x_1)\},\ 0<\theta<1.$$

But since $f'(x_0)$ is continuous at x_0, $f'\{x_1+\theta(x_2-x_1\}$ tends to $f'(x_0)$ when $\theta_1, \theta_2\to$P, *i.e.*, when x_1 and $x_2\to x_0$, and the limit of the chord is $y-f(x_0)=f'(x_0)(x-x_0)$, that is the tangent at P.]

21. Prove that the equation of the tangent at any point $(4m^2, 8m^3)$ of the curve $y^2=x^3$ is $y=3mx-4m^3$ and that it meets curve again at the point $(m^2, -m^3)$. Show that if $9m^2=2$, the tangent is also a normal of the curve.

22. Tangents are drawn from the origin to the curve $y=\sin x$. Prove that their points of contact lie on $x^2y^2=x^2-y^2$.

23. If the line $lx+my=1$ touch the curve

$$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=1,$$

prove that

$$(al)^{\frac{n}{n-1}}+(bm)^{\frac{n}{n-1}}=1.$$

24. Prove that all the curves represented by the equation

$$\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$$

for different values of n, have a common tangent at the point (a, b).

25. If $Ax+By=C$ is a normal to the curve $a^{n-1}y=x^n$, show that

$$n^nA^nC^{n-1}=a^{-1}B\,(nA^2+B^2)^{n-1}.$$

26. Show that the angles of intersection of the parabolas $y^2=4ax$ and $x^2=4by$ are

$$\frac{\pi}{2} \text{ and } \tan^{-1}\frac{3a^{\frac{1}{3}}b^{\frac{1}{3}}}{2(a^{\frac{2}{3}}+b^{\frac{2}{3}})}.$$

8. Tangents at the origin.

The equation of the tangent *at the origin* to a curve *which passes through the origin* is by the general formula

$$x\left(\frac{\partial f}{\partial x}\right)_{\substack{x=0\\y=0}} + y\left(\frac{\partial f}{\partial x}\right)_{\substack{x=0\\y=0}} = 0.$$

This requires us to find the values of $\partial f/\partial x$ and $\partial f/\partial y$ at the origin which becomes somewhat complicated in the case of higher degree curves. We give below a method which will enable us to find readily such tangents by inspection.

Case (*i*). Suppose that the absolute term is zero but the first degree terms are present.

Let the equation of the curve be

$$u_n + u_{n-1} + \ldots\ldots + (ax^2 + hxy + by^2) + gx + fy = 0,$$

and let the line $y = mx$ through the origin cut the curve in P, Q, R, S.....

The abscissae of the points of intersection of the line $y = mx$ with the curve will be obtained by substituting mx for every y in the equation to the curve.

Let the result of substitution be

$$v_n + v_{n-1} + \ldots + x^2(a + hm + bm^2) + x(g + fm) = 0,$$

which is an equation of the nth degree in x only.

One root of this is obviously zero, since origin is one of the points of intersection.

A second root will be zero, two of the points of intersection being coincident at the origin if

$$g + fm = 0, \text{ } i.e., \text{ if } m = -g/f.$$

In this case, the line $y = mx$ will become $y = -\frac{g}{f}x$ or

$gx + fy = 0$, which is tangent at the origin.

Case (*ii*). If the first degree terms are also absent from the equation of the curve, we have $g = 0$ and $f = 0$, the equation of the curve is then of the form

$$u_n + u_{n-1} + \ldots\ldots + (ax^2 + hxy + by^2) = 0.$$

On substituting mx for y, we get

$$v_n + v_{n-1} + \ldots\ldots + x^2(a + hm + bm^2) = 0.$$

Two roots are zero, so that every line through the origin cuts the curve in two points thereat.

A third root will be zero if

$$a+hm+bm^2=0.$$

In this case there will be two values of m and the equations of the two tangents at the origin will be obtained by eliminating "m" between $y=mx$ and $a+hm+bm^2=0$.

The result of elimination is $a+h\frac{y}{x}+b\left(\frac{y}{x}\right)^2=0$

i.e., $ax^2+hxy+by^2=0$, which is the joint equation of the two tangents at the origin.

The rule can be easily extended.

Thus we get the following rule for finding the tangent or tangents at the origin to a curve passing through the origin.

"*The terms of the lowest degree in x and y equated to zero give the equation of the tangent or tangents (real or imaginary) at the origin.*"

EXAMPLES XXVI

Find the tangent or the tangents at the origin in the following curves:

1. $x^3=6y^2$. 2. $y^2=ax$. 3. $x^2=ay$. 4. $y^4=a^2x^2$.
5. $y^2(3x+2y)=4y^2+9x^2$. 6. $x^2(2a-y)=y^3$.
7. $x^3+y^3+3x-2y=0$. 8. $x^3-6y^2+y^3=0$.
9. $x^2y^2=a^2(x^2-y^2)$. 10. $x^2(x-y)+a^2(x+y)=0$.
11. $x^3+y^3=3axy$. 12. $x^4+y^4-2axy^2-3a^2x^3=0$.

Geometrical Results.

9. Intercepts of tangents on the axes.

If in the equation of the tangent

$Y-y\frac{dy}{dx}(X-x)$, we put $Y=0$, then

$X=x-y\left/\frac{dy}{dx}\right.$; this is, therefore, the value of OT.

Thus the tangent cuts off an intercept on the x-axis $=x-y/y$.

Note. The student should observe that we are really finding the abscissa of the point of intersection of tangent with $Y=0$ (x-axis), *i.e.*, we are solving the simultaneous equations

(*i*) $Y-y=dy/dx.(X-x)$, (*ii*) $Y=0$

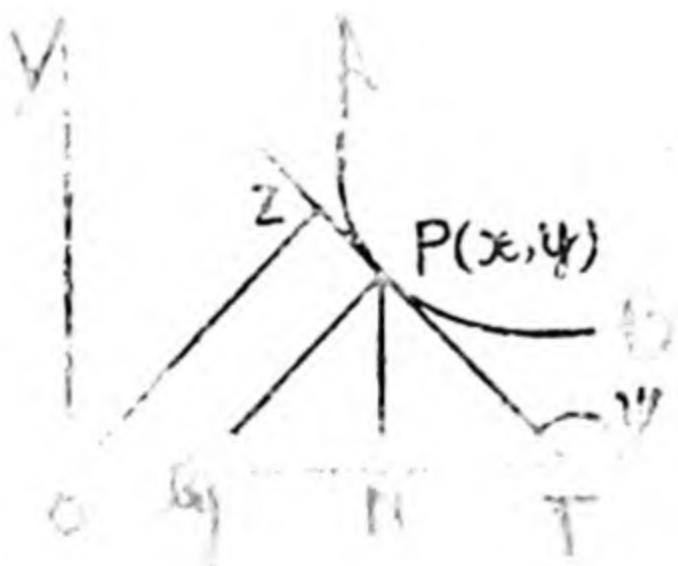
in order to determine the value of X.

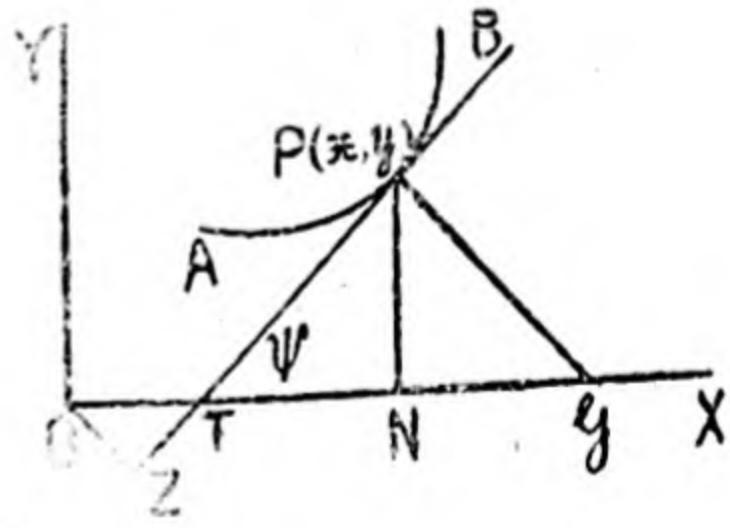
Again, if we put $X=0$, we get $Y=y-x\frac{dy}{dx}$; this is, therefore, the value of Ot. Thus the tangent cuts off an intercept on y-axis $=y-x/y'$.

In a similar manner, the intercepts of the normal on the axes can be easily found.

10. Lengths of subtangent and subnormal.

Let P be a point (x, y) on the curve APB whose equation is $y=f(x)$. Let the tangent PT and the normal PG meet the x-axis in T and G and and let ψ be the angle that tangent makes with the x-axis so that $\tan\psi=\frac{dy}{dx}$ and therefore,





$$\sin\psi=\frac{dy}{dx}\Big/\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Then, *length of tangent* $=PT=\sqrt{PN^2+TN^2}$

$$=\sqrt{PN^2+PN^2\cot^2\psi}=\sqrt{y^2+\frac{y^2}{\left(\frac{dy}{dx}\right)^2}}$$

$$= y\sqrt{1+\frac{1}{y'^2}} = \frac{y}{y'}\sqrt{1+y'^2}$$

where $y' = \frac{dy}{dx}$.

Length of normal $= PG = \sqrt{PN^2 + NG^2}$

$$= \sqrt{y^2 + y^2 \tan^2\psi} = y\sqrt{1+y'^2}.$$

Subtangent = *projection of length of the tangent* on the x-axis $= TN = PN \cot\psi = y/y'$.

Subnormal = *projection of the length of the normal* on the x-axis $= NG = PN \tan\psi = yy'$.

Note. We suppose x to be always increasing towards the right; y may, then, increase and decrease as x moves in the positive direction to the right. If y decreases with x, dy/dx is positive; if y decreases with x, dy/dx is negative. Hence it follows that *subtangent* is measured from T to the right or to the left according as y/y', is + or −; and *subnormal* is measured from N to the right or to the left according as yy' is + or −.

11. Perpendicular from the origin on the tangent.

Let p be the length of the perpendicular.

Then $p = OZ = OT \sin\psi = (ON + NT)\sin\psi$

$$= \{x + (-y/y')\}.\ y'\sqrt{1+y'^2}$$

$$= (xy' - y)/\sqrt{1+y'^2}.$$

Otherwise: p being the perpendicular from $(0, 0)$ on the tangent

$(X-x)\frac{\partial f}{\partial x} + (Y-y)\frac{\partial f}{\partial y} = 0$, at the given point (x, y), p is given by the equation

$$p = \frac{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}}{\sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}}.$$

Ex. 1. Show that the subtangent at any point of the logarithmic curve $y=ce^{\frac{x}{a}}$ is constant. Find also the length of the subnormal.

Differentiating logarithmically

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{a} \qquad \therefore \frac{dy}{dx}=\frac{y}{a}.$$

Hence subtangent at $(x, y)=y/\frac{dy}{dx}=y/\frac{y}{a}$

$=a$, which is constant.

Subnormal $=y\frac{dy}{dx}=y.\frac{y}{a}=\frac{y^2}{a}$.

It is the only curve in which the subtangent is constant.

Ex. 2. Find the length of perpendicular from origin, length of tangent and length of normal in the curve of Ex. 1.

Equation of tangent is $Y-y=\frac{y}{a}(X-x)$

i.e., $yX-aY+y(a-x)=0$

(*i*) $\therefore$ p=perpendicular from origin (0, 0)

$$=\frac{y(a-x)}{\sqrt{y^2+a^2}}.$$

(*ii*) Length of tangent $=y\sqrt{1+y^2/y'}$

$$=\frac{y}{a}\sqrt{a^2+y^2}\times\frac{a}{y}=(a^2+y^2)^{\frac{1}{2}}$$

Normal-length $=y\sqrt{1+\left(\frac{dy}{dx}\right)^2}=y\sqrt{1+\frac{y^2}{a^2}}$

$$=\frac{y}{a}(a^2+y^2)^{\frac{1}{2}}.$$

Ex. 3. Show that the subnormal at any point of a parabola is of constant length.

Let $y^2=4ax$ be the parabola. Differentiating with respect to x, we get,

$$2y\frac{dy}{dx}=4a, \quad \because \frac{dy}{dx}=\frac{2a}{y}.$$

Hence subnormal at $(x, y) = y\frac{dy}{dx} = y\times\frac{2a}{y}=2a,$

which is constant.

It may be pointed out that the only curve for which the subnormal is constant is a parabola.

EXAMPLES XXVII

1. Find the length and equation of the tangent and the normal at $\theta=\frac{\pi}{2}$ to the curve

$$x=a(\theta-\sin\theta),\ y=a(1-\cos\theta).$$

2. Show that in the hyperbola $xy=m^2$ the product of the intercepts of the tangent is constant.

3. Show that in the curve $ay^3=x^3$ the square of the sub-tangent varies as the normal.

4. Prove that the sum of the intercepts of the tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ on the co-ordinate axes is constant.

5. Show that the portion of the tangent to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ which is intercepted between the axes is of constant length.

6. In the curve $x^my^n=a^{m+n}$ show that the portion of the tangent intercepted between the axes is divided in a constant ratio at the point of contact.

Examine the case when

(*i*) $m=-1, n=2$. (*ii*) $m=1, n=1$.

7. Show that the tangent to the curve

$$\frac{x+\sqrt{a^2-y^2}}{a}=\log\frac{a+\sqrt{a^2-y^2}}{y}$$

is constant and equal to a.

8. In the curve $x=a\left(\cos t+\log \tan \frac{t}{2}\right)$, $y=a \sin t$ show that the portion of the tangent between the point of contact and the axes of x is of constant length.

9. Prove that the perpendicular drawn from the foot of the ordinate to the tangent of a curve is $y/(1+y'^2)^{\frac{1}{2}}$.

10. Prove that for the catenary $y=c \cos h \frac{x}{c}$ the length of perpendicular from the foot of the ordinate on the tangent is of constant length.

11. In the tractory $x=\sqrt{c^2-y^2}+\frac{c}{2} \log \frac{c-\sqrt{c^2-y^2}}{c+\sqrt{c^2-y^2}}$, prove that the portion of the tangent intercepted between the point of contact and the axis of x is of constant length.

Polar Co-ordinates

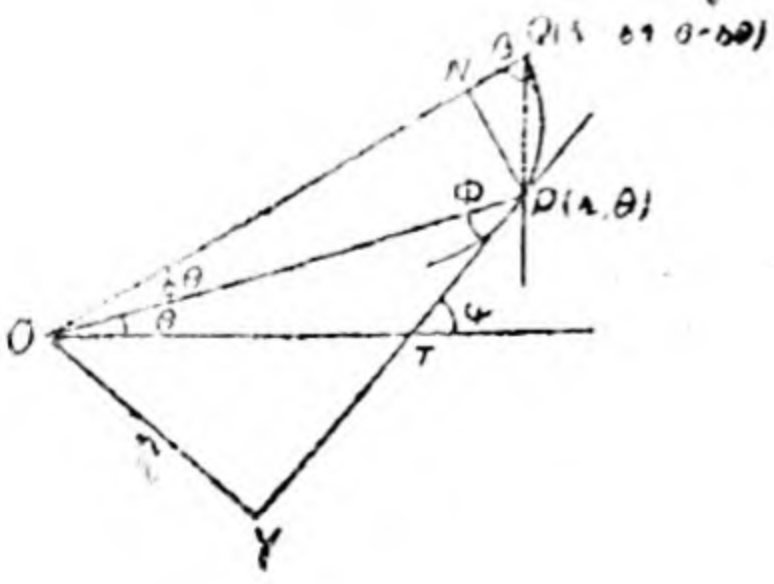
12. Angle betweent tangent and radius vector

We have seen that there is a simple geometrical interpretation for dy/dx. Though there is no simple geometrical meaning for $dr/d\theta$, but there is an important formula connecting $dr/d\theta$ with the angle between the tangent and the radius vector to the point of contact as we shall presently see.

Let $r=f(\theta)$ or $f(r, \theta)=0$ be the equation of any curve in polar co-ordinates. Let P be the point (r, θ) and let a neighbouring point Q be $(r+\delta r, \theta+\delta\theta)$. Draw $PN \perp OQ$ and let the angle between the radius vectar OP and the tangent PT at P be denoted by ϕ and let ψ be the angle the tangent makes with the positive direction of the initial line. Let $\angle PQN=\beta$.

We shall prove that

$$\tan\phi = r\, d\theta/dr.$$



We have

$$PN=r \sin \delta\theta$$
$$ON=r \cos \delta\theta$$
$$NQ=OQ-ON$$
$$=(r+\delta r)-r \cos \delta\theta$$
$$=\delta r+r(1-\cos \delta\theta).$$

Then $\tan PQN=\frac{PN}{NQ}$

$$= \frac{r \sin \delta\theta}{\delta r + r(1 - \cos \delta\theta)}$$

$$= \frac{r\dfrac{\sin \delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + \dfrac{2r \sin^2 \dfrac{\delta\theta}{2}}{\delta\theta}}$$

$$i.\ e.,\ \tan \beta = \frac{\dfrac{r \sin \delta\theta}{\delta\theta}}{\dfrac{\delta r}{\delta\theta} + \dfrac{r \sin \dfrac{\delta\theta}{2}}{\dfrac{\delta\theta}{2}} \cdot \sin \dfrac{\delta\theta}{2}}.$$

As Q moves closer and closer to P and tends to coincidence with P, $\delta\theta$ tends to 0 and we have

$$\textit{limit of left side} = \text{Lt.} \tan \beta = \tan \phi,$$

$$\textit{and limit of right side} = \frac{r}{\dfrac{dr}{d\theta}}$$

$$\therefore \lim_{\delta\theta \to 0} \frac{\sin \dfrac{\delta\theta}{2}}{\dfrac{\delta\theta}{2}} = 1 \quad \text{and} \quad \lim_{\delta\theta \to 0} \sin \frac{\delta\theta}{2} = 0$$

$$\therefore \tan \phi = \frac{r}{\left(\dfrac{dr}{d\theta}\right)} = r\frac{d\theta}{dr}.$$

Note. ϕ is geometrically identified with the angle between the prolongation of the radius vector (which is its positive direction) and the positive direction of the tangent which is the direction in which arc s (or θ) increases.

13. Let p denote the perpendicular from the pole O on the tangent at P.

To prove that $\dfrac{1}{p^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$.

From the figure we have

$$p = r \sin \phi$$

$$\therefore \quad \frac{1}{p^2} = \frac{1}{r^2 \sin^2 \phi} = \frac{1}{r^2}(1 + \cot^2 \phi)$$

$$= \frac{1}{r^2}\left\{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}$$

$$= \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

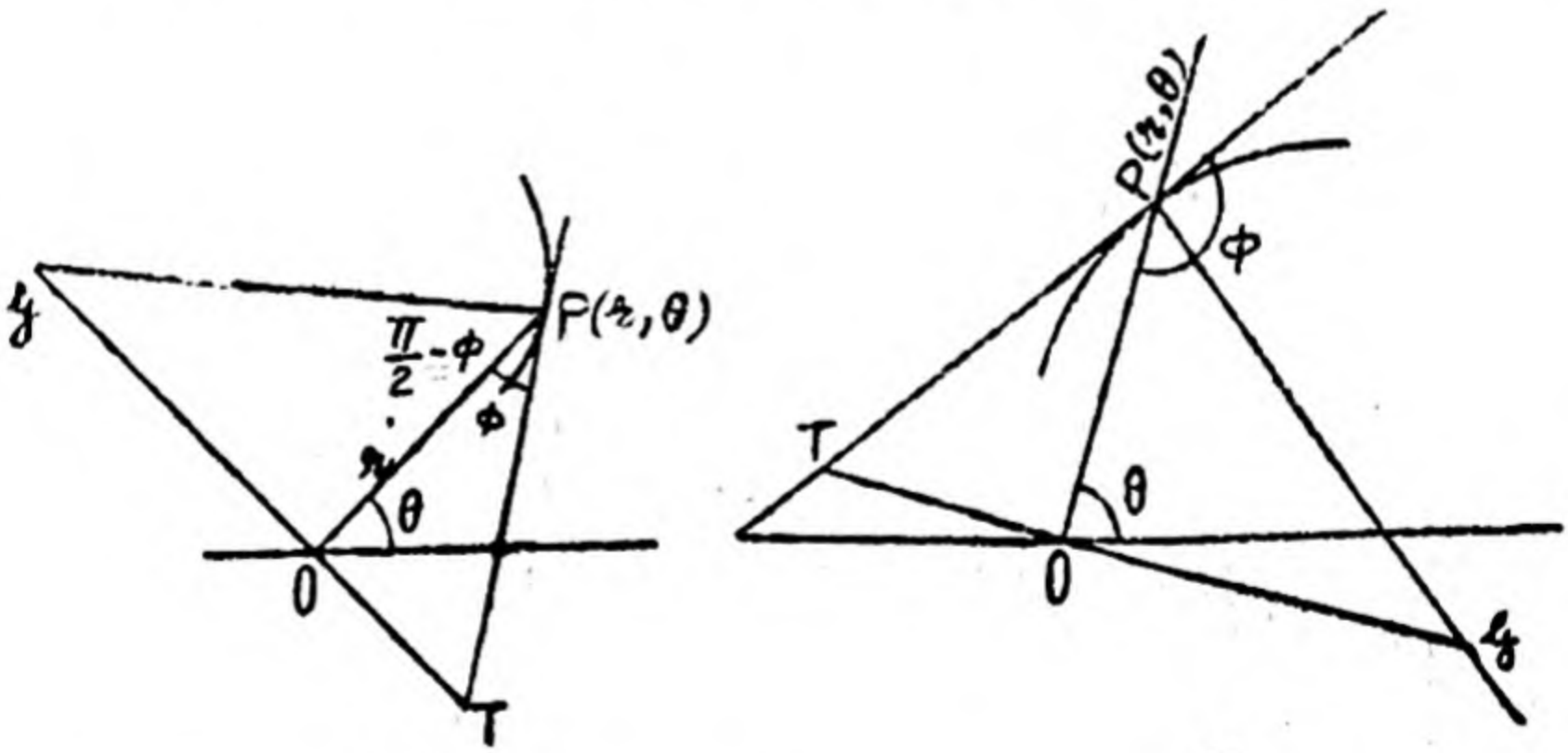
Let $u = \frac{1}{r}$; differentiating with respect to θ,

$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}.$$

Hence, introducing u and $\frac{du}{d\theta}$ in the expression for $\frac{1}{p^2}$, we get $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

14. Polar Sub-tangent and Polar Sub-normal.

Let P be the point (r, θ). Through the origin O, let GOT



be drawn perpendicular to the radius vector OP, meeting the tangent in T and the normal in G. The projections of the tangent PT and of the normal PG on this line are called the *polar subtangent* and *polar subnormal*. Then

$$\text{OT}=\textit{polar sublangent}=r \tan \phi=r^2 \frac{d\theta}{dr}$$

$$\text{OG}=\textit{polar sub-normal}=r \cot \phi=r.\frac{dr}{rd\theta}=\frac{dr}{d\theta}.$$

Note. The initial line of θ is OX and θ is measured positive in the counter-clockwise direction. The direction of the increase of s is so chosen that s and θ increase together the angle ϕ is reckoned positive counter clockwise from the positive radius vector to the positive direction of the tangent.

We then have $\psi=\theta+\phi$

If $u=\frac{1}{r}$, $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$.

Hence *polar sub-tangent*$=\text{OT}=r^2\frac{d\theta}{dr}=-\frac{d\theta}{du}$.

Ex. 1. Prove that in the curve $r=ae^{\theta \cot \alpha}$ (a and α are constants) the angle between the tangent and the radius vector is the same at all points of the curve. Find the polar subtangent, the polar subnormal and the perpendicular from the origin.

Taking logarithms, we get

(*i*) $\log r=\log a+\theta \cot \alpha$.

Differentiating with respect to θ,

$$\frac{1}{r}\frac{dr}{d\theta}=\cot \alpha, \textit{ i.e.,} \cot \phi=\cot \alpha$$

$\phi=\alpha$ (a constant which is independent of r and θ, *i.e.*, of the position of the point on the curve).

On account of this property, the curve is called the equi-angular spiral.

(*ii*) The polar subtangent$=r^2\frac{d\theta}{dr}=r\,r\frac{d\theta}{dr}=r \tan \alpha$.

(*iii*) The polar subnormal$=\frac{dr}{d\theta}=r \cot \alpha$.

(*iv*) $p=r \sin \phi=r \sin \alpha$.

Ex. 2. In the cardiod $r=a(1-\cos\theta)$ prove that

$$(i)\ \phi=\frac{\theta}{2} \qquad (ii)\ p=2a\sin^3\frac{\theta}{2} \qquad (iii)\ 2ap^2=r^3.$$

(*i*) Differentiating the equation of the curve with respect to θ, we get

$$\frac{dr}{d\theta}=+a\sin\theta$$

$$\therefore\ r\frac{d\theta}{dr}=\frac{r}{a\sin\theta}=\frac{a(1-\cos\theta)}{a\sin\theta}=\tan\frac{\theta}{2},$$

$$\therefore\ \tan\phi=\tan\theta/2, \qquad \text{or } \phi=\theta/2.$$

$$(ii)\ p=r\sin\phi=r\sin\frac{\theta}{2}$$

But $r=2a\sin^2\dfrac{\theta}{2}$ $\qquad\therefore\ p=2a\sin^3\dfrac{\theta}{2}$

$$(iii)\ p=2a\sin^3\frac{\theta}{2}=2a\left(\frac{r}{2a}\right)^{\frac{3}{2}}.$$

Squaring and simplifying, we get the result.

Ex. 3. In the curve $r^2=a^2\cos 2\theta$, show that $p=\dfrac{r^3}{a^2}$.

Differentiating $r^2=a^2\cos 2\theta$ *w. r. t.* θ, we have

$$r\frac{dr}{d\theta}=-a^2\sin 2\theta \quad\therefore\quad \frac{dr}{d\theta}=\frac{-a^2\sin 2\theta^2}{r}.$$

$$\text{Hence } \frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\times\frac{(-a^2\sin 2\theta)^2}{r^2}=\frac{r^4+a^4\sin^2 2\theta}{r^6}$$

$$=\frac{a^4\cos^2 2\theta+a^4\sin^2 2\theta}{r^6}=\frac{a^4}{r^6}$$

$$\therefore\qquad p=\frac{r^3}{a^2}.$$

15. Pedal Equation or " p, r equation " of a curve.

If r be the radius vector to any point of a curve and p denotes the perpendicular from the origin to the tangent at that point, then the relation between r and the corresponding p is called the *pedal equation of the curve.*

In Ex. 2 and Ex. 3 above we have found the pedal equation of the curve $r=a(1-\cos\theta)$ and $r^2=a^2\cos 2\theta$ respectively.

16. To *find the pedal equation* (with respect to the pole) *of a curve whose polar equation is given.*

Let $f(r, \theta)=0$ be the polar equation of the curve, then

$$p=r\sin\phi \tag{1}$$

and

$$\tan\phi=r\frac{d\theta}{dr} \tag{2}$$

The co-ordinates r and θ satisfy (1) and (2) and also $f(r, \theta)=0$.

Eliminating ϕ and θ from these equations, we get a relation between p and r which is the required pedal equation.

Alternative method. The p, r relation in this case may also be obtained by eliminating θ between the equations

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^1}\left\{\frac{dr}{d\theta}\right\}^2 \text{ and } f(r, \theta)=\theta$$

Ex. 1. Find the pedal equation of $r^m=a^m\sin m\theta$.

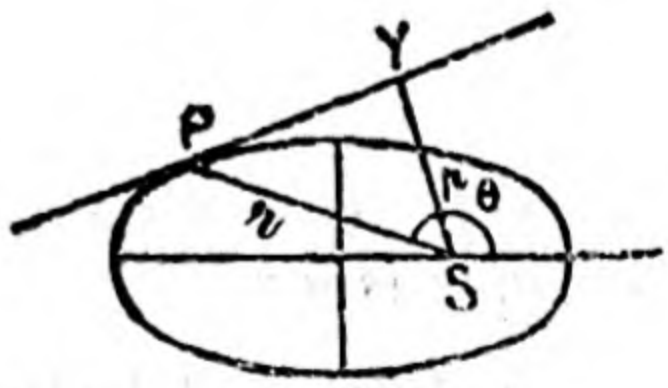
Taking logarithms, $m\log r=m\log a+\log\sin m\theta$.
Differentiating *w. r. t.* θ.

$$\frac{mdr}{rd\theta}=m\cot m\theta, \text{ i.e., } \cot\phi=\cot m\theta\ ;\ \therefore\ \phi=m\theta.$$

$$\because\ p=r\sin\phi=r\sin m\theta=r\frac{r^m}{a^m},\ \therefore\ p.a^m=r^{m+1}.$$

Ex. 2. Find the pedal equation of the ellipse

$$\frac{l}{r}=1+e\cos\theta.$$



$$lu=1+e\cos\theta$$

$$\therefore\ \frac{du}{d\theta}=\frac{-e\sin\theta}{l}$$

$$\therefore\ \frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2=\left(\frac{1+e\cos\theta}{l}\right)^2+\left(\frac{-e\sin\theta}{l}\right)^2$$

$$=\frac{1}{l^2}(1+e^2+2e\cos\theta)=\frac{1}{l^2}\left\{1+e^2+2\left(\frac{l}{r}-1\right)\right\}$$

$$= \frac{1}{l^2}\left\{\frac{2l}{r}(1-e^2)\right\}$$

$$= \frac{1}{l^2}\left(\frac{2l}{r} - \frac{l}{a}\right) = \frac{1}{l}\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$\left[\because l = \frac{b^2}{a} = \frac{a^2(1-e^2)}{2} = a(1-e^2)\right]$$

$\therefore \quad \frac{1}{p^2} = \frac{1}{l}\left(\frac{2}{r} - \frac{1}{a}\right)$ which is the pedal equation of the ellipse focus being the pole.

Alternative method :

Let SP$=r$. S'P$=r'$, where S, S' are the real foci and P any point of an ellipse (semi-axes a, b). Let p and p' be the perpendiculars from S, S' on the tangent at P. Then since SP, S'P are equally inclined to the tangent at P,

(*i*) $p = r \sin\phi$ (*ii*) $p' = r' \sin\phi$ (*iii*) $pp' = b^2$

(*iv*) $r + r' = 2a$.

Eliminate ϕ, p' r'. $\frac{2a}{r} - 1 = \frac{r'}{r}$ from (*iv*)

$$= \frac{p'}{p} \text{ [from (i), (ii)]}$$

$$= \frac{b^2}{p^2}$$

i.e., $$= \frac{b^2}{p^2} = \frac{2a}{r} - 1.$$

***17.** To find the *pedal equation of a curve whose cartesian equation is given.*

Let $f(x, y)=0$ be the cartesian equation of the curve; the equation of the tangent at (x, y) is

$$X\frac{\partial f}{\partial x} + Y\frac{\partial f}{\partial y} - \left\{x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y}\right\} = 0,$$

and the perpendicular p from (0, 0) on the tangent is given by

$$p^2=\frac{\left\{x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}\right\}^2}{\left\{\frac{\partial f}{\partial x}\right\}^2+\left\{\frac{\partial f}{\partial y}\right\}^2} \qquad \ldots(1)$$

The radius vector to (x, y) is given by

$$x^2+y^2=r^2 \qquad \ldots(2)$$

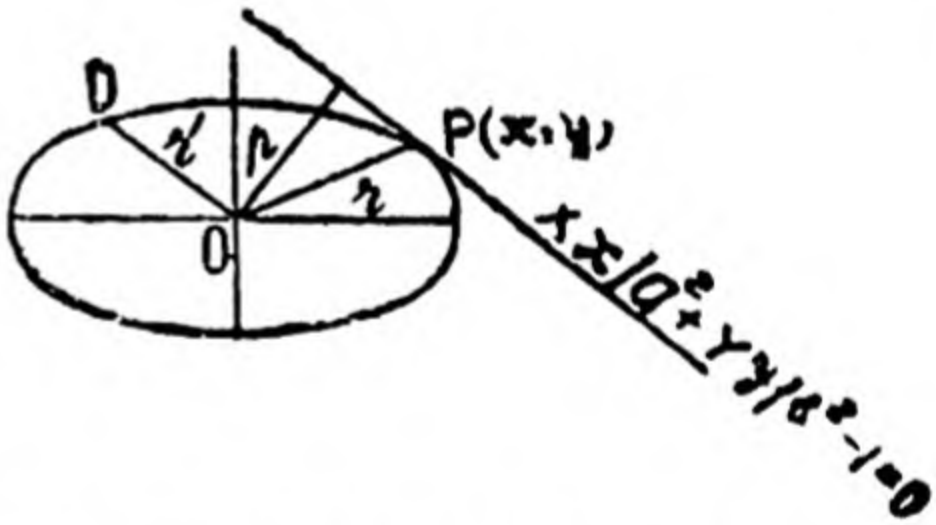
The co-ordinates (x, y) satisfy equations (1) and (2) and $f(x, y)=0$. $\ldots(3)$

If we eliminate x and y between equations (1), (2) and (3), we shall get a relation between p and r which is the required pedal equation.

Ex. Find the pedal equation, with respect to its centre of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. $\ldots(1$

We have

$$p^2=\frac{1}{\left(\frac{x}{a^2}\right)^2+\left(\frac{y}{b^2}\right)^2}$$



i.e., $\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{p^2}$ $\ldots(2)$

and $x^2+y^2=r^2$ $\ldots(3)$

Eliminating x^2 and y^2 between (1), (2) and (3) we get :—

$$\begin{vmatrix} 1 & \frac{1}{b^2} & 1 \\ a^2 & & \\ \frac{1}{a^4} & \frac{1}{b^4} & \frac{1}{p^2} \\ 1 & 1 & r^2 \end{vmatrix}=0.$$

i.e., $\frac{1}{a^4}\left(\frac{r^2}{b^4}-\frac{1}{p^2}\right)+\frac{1}{b^2}\left(\frac{1}{p^2}-\frac{r^2}{a^4}\right)+\left(\frac{1}{a^4}-\frac{1}{b^4}\right)=0$

i.e., $(b^4-a^4)-(b^2-a^2)r^2=\frac{a^2b^2}{p^2}=(b^2-a^2)$.

Hence we get, after removing factor b^2-a^2,

$(a^2+b^2)-r^2=\frac{a^2b^2}{p^2}$, which is the *pedal equation of the ellipse, centre being the pole.*

Alternative geometrical method. The pedal equation may also be derived from the known geometrical properties of the ellipse. We have

$$(i)\ \mathrm{CP}^2+\mathrm{CD}^2=a^2+b^2$$

$$(ii)\ \ \mathrm{CD}.p=ab\ ;$$

whence eliminating CD, we get

$$a^2+b^2-\mathrm{CP}^2=\frac{a^2b^2}{p^2},\ \ i.\ e.,\ a^2+b^2-r^2=\frac{a^2b^2}{p^2}.$$

EXAMPLES XXVIII

1. For the parabola $\frac{2a}{r}=1-\cos\theta$, show that

   $(i)\ \phi=\pi-\frac{\theta}{2}$ (ii) polar subtangent $=2a\cos\theta$

   $(iii)\ p^2=ar.$

2. For the cardiode $r=a(1-\cos\theta)$ show that

   $(i)\ \phi=\frac{\theta}{2}$ $(ii)\ p=2a\sin^3\frac{\theta}{2}$ $(iii)\ 2ap^2=r^3.$

3. In the curve $r=a\,\theta$, show that

   (i) $\tan\phi=\theta.$ (ii) the polar subnormal is constant.

4. In the curve $r\theta=a$, show that

   (i) $\tan\phi=-\theta$ (ii) the polar subtangent is constant.

5. Find the angle of intersection of the curves

   $r=a(1+\cos\theta)$ and $r=b\ (1-\cos\theta)$.

6. For the curve $r^n=a^n\cos n\theta$ show that $pa^n=r^{n+1}$.

   What is the corresponding result for $r^n\sin n\theta$?

7. In the curve $r=ae^{\theta \cos \alpha}$ show that $p=r \sin \alpha$.

8. Show that in the curve

$$\theta=\frac{\sqrt{r^2-a^2}}{a}-\cos^{-1}\frac{a}{r},\ \cos\phi=\frac{a}{r}.$$

9. Prove that $\dfrac{d\phi}{d\theta}=\dfrac{u\dfrac{d^2u}{d\theta^2}=\left(\dfrac{du}{d\theta}\right)^2}{u^2+\left(\dfrac{du}{d\theta}\right)^2}$

[*Hint* $\because\ \phi=\tan^{-1}\dfrac{r}{\dfrac{du}{d\theta}}=\tan^{-1}\left(-\dfrac{u}{\dfrac{du}{d\theta}}\right)$

10. In the curve

$$x=(a+b)\cos\ \theta-b\ \cos\frac{a+b}{b}\theta$$

$$y=(a+b)\sin\ \theta-b\ \sin\frac{a+b}{b}\theta.$$

show that

$$p=(a+2b)\sin\frac{a\theta}{2\theta},\ \psi=\frac{a+2b}{2b}\theta,\ p=(a+2b)\sin\frac{a\psi}{a+2b}$$

and $(r^2-a^2)(a+2b)^2=4b(a+b)p^2$.

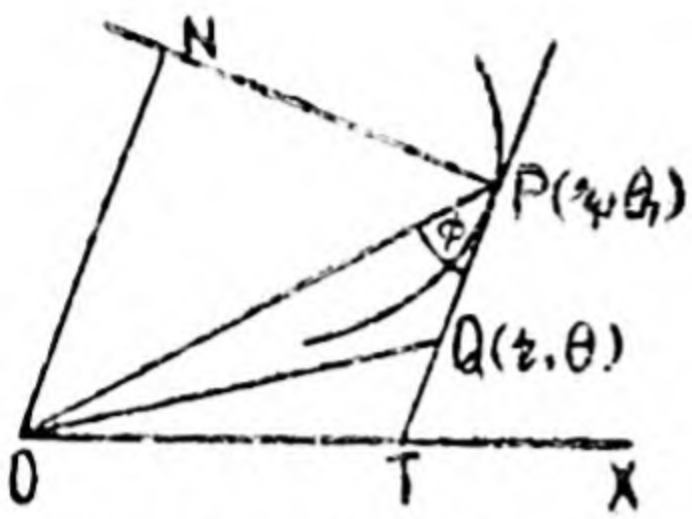
11. If ϕ be the angle which the tangent to a curve makes with the radius vector drawn from the origin, prove that

$$\tan\phi=\frac{rd\theta}{dr}=\frac{x\dfrac{dy}{dx}-y}{x+y\dfrac{dy}{dx}}.$$

12. Find the pedal equation of the circle (*i*) $x^2+y^2=2ax$ (*ii*) parabola $y^2=4a(x+a)$ origin being the pole.

13. *Find the equation of the tangent and the normal at any point* (r_1, θ_1) *on the curve* $r=f(\theta)$.

[(*i*) Let PT be the tangent to the curve at any given point $P(r_1, \theta_1)$. Let (r, θ) be the co-ordinates of any point Q on the tangent. Let $\angle OPT=\phi$.



From $\triangle$ OPQ, we have

$$\frac{r_1}{r}=\frac{\sin \text{OQP}}{\sin \text{QPO}}=\frac{\sin(\theta_1-\theta+\phi)}{\sin\phi}$$

$$=\sin(\theta_1-\theta)\cot\phi+\cos(\theta_1-\theta)$$

Hence $$\frac{r_1}{r}=\sin(\theta_1-\theta).\frac{1}{r_1}\frac{dr_1}{d\theta_1}+\cos(\theta_1-\theta) \quad \ldots(1)$$

This is the relation between any r,θ and is therefore, the *polar equation of the tangent.*

If we put $u=\frac{1}{r}$. $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$ and the equation of tangent will take the form

$$u=u_1\cos(\theta-\theta_1)+\frac{du_1}{d\theta_1}\sin(\theta-\theta_1).$$

(*ii*) Normal at (r_1, θ) to the curve $r=f(\theta)$.

Again let N(r, θ) be any point on the normal PN ; so that ON$=r$ and XON$=\theta$.

From $\triangle$ OPN, we have

$$\frac{\text{OP}}{\text{ON}}=\frac{\sin \text{PNO}}{\sin \text{OPN}}=\frac{\sin\left(\theta-\theta_1+\frac{\pi}{2}-\phi\right)}{\sin\left(\frac{\pi}{2}-\phi\right)}$$

Hence $$\frac{r_1}{r}=\sin(\theta-\theta_1)\tan\phi+\cos(\theta-\theta_1)$$

$$=\sin(\theta-\theta_1).\,r_1\frac{d\theta_1}{dr_1}+\cos(\theta-\theta_1) \quad \ldots(2)$$

This is a relation between the co-ordinates r, θ of any point on the normal and is, therefore, the polar equation of the normal to a curve at point (r_1, θ_1).

Note. The equation (2) may also be written as

(*i*) $$r_1\frac{dr_1}{d\theta_1}/r=\frac{d}{d\theta_1}\{r_1\cos(\theta-\theta_1)\}$$

(*ii*) $u=u_1 \cos(\theta-\theta_1)-u_1^2\frac{d\theta_1}{du_1}\sin(\theta-\theta_1)$,

where $u=\frac{1}{r}$].

14*. Find the polar equations of the tangent and normal at the point (r_1, θ_1) to the following curves :

(*i*) $\frac{l}{r}=1+e\cos\theta$. (*ii*) $r=a(1+\cos\theta)$

(*iii*) $r^2=a^2\cos 2\theta$. (*iv*) $r=a\sin 2\theta$.

15*. Find the pedal equation of (*i*) an ellipse (*ii*) a hyperbola (*iii*) a parabola, focus being the pole in each case.

16.* Show that the locus of the extremity of the polar subnormal of the curve $r=\phi(\theta)$ is $r=\phi'\left(\theta-\frac{\pi}{2}\right)$.

Hence show that the locus of the extremity of the polar subnormal of the equiangular spiral $r=e^{m\theta}$ is another equiangular spiral.

17.* Show that the locus of the extremity of the polar subtangent of the curve $u=f(\theta)$ is $u+f'\left(\frac{\pi}{2}+\theta\right)=0$.

Deduce that the locus of the extremity of the polar subtangent of the curve

$$r\left(m+n\tan\frac{\theta}{2}\right)=1+\tan\frac{\theta}{2} \text{ is}$$

$$\frac{m-n}{2}r=1+\cos\theta.$$

18.* Show that the points of contact of tangents from the point (h, k) to the curve $x^3+y^3=3axy$ lie on the conic

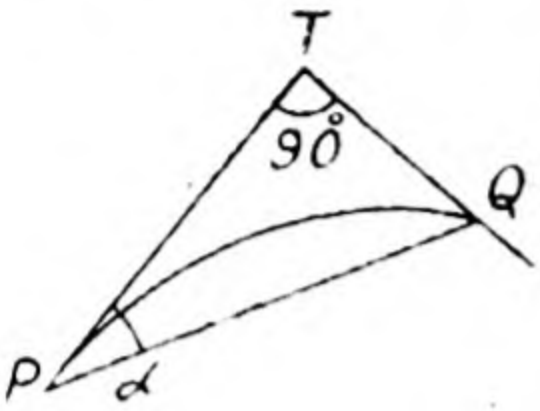
$$hx^2-axy+ky^2=a^2(kx+hy).$$

Derivative of the Arc

18.* Let PQ be an arc of any curve supposed to be everywhere concave towards its chord.

Then $\lim_{Q\to P}\frac{\text{arc PQ}}{\text{chord PQ}}=1$.

Draw QT perpendicular to PT the tangent to the arc at P. Let QT not meet the arc in any point other than Q ; this is secured by taking Q near enough to P.



Let $\angle QPT=\alpha$.

Then $PT+QT>$ arc PQ, since it may be taken as axiomatic that the length of any arc which is everywhere concave towards its chord is less than the length of any broken line enclosing it and having the same extremities.

Also arc PQ > chord PQ.

Hence $PT+TQ>$ arc $PQ>$ chord PQ,

i. e., $PQ\cos\alpha+PQ\sin\alpha>$ arc $PQ>PQ$.

i.e., $\cos\alpha+\sin\alpha>\dfrac{\text{arc PQ}}{\text{chord PQ}}>1.$

As Q travels along the arc and tends to coincidence with P, $\alpha\to0$, $\sin\alpha\to0$ and $\cos\alpha\to1$ so that

$$\text{Lt}\ (\cos\alpha+\sin\alpha)=1.$$

Hence $\underset{Q\to P}{\text{Lt}}\ \dfrac{\text{arc PQ}}{\text{chord PQ}}=1$, *i.e.*, $\underset{\delta s\to o}{\text{Lt}}\ \dfrac{\delta\beta}{\delta c}=1$.

The following proof of this inequality is, however, worthy of consideration.

Let PM, MN, NQ, be the *broken line* standing on the same chord as arc PQ. Then from the figure below we have

$PT+TT_1>PT_1$ *i.e.*, $>PM+MT_1$

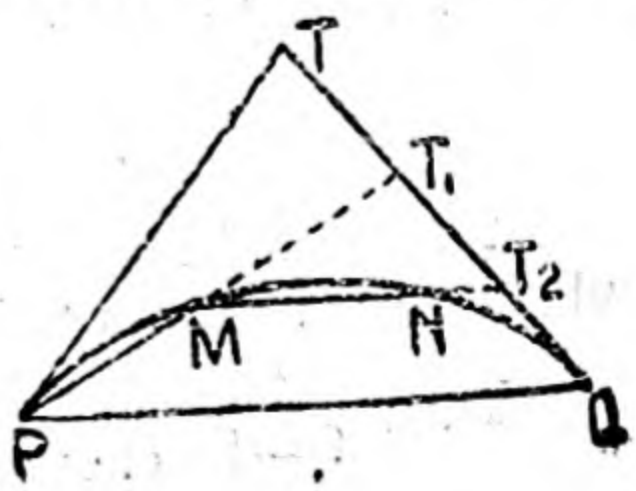
$MT_1+T_1T_2>MT_2$, *i.e.*, $>MN+NT_2$

$NT_2+T_2Q>NQ$

adding and removing common lengths from the two sides we get

$PT+TQ>$ *broken line*, $PM+MN+NQ$.

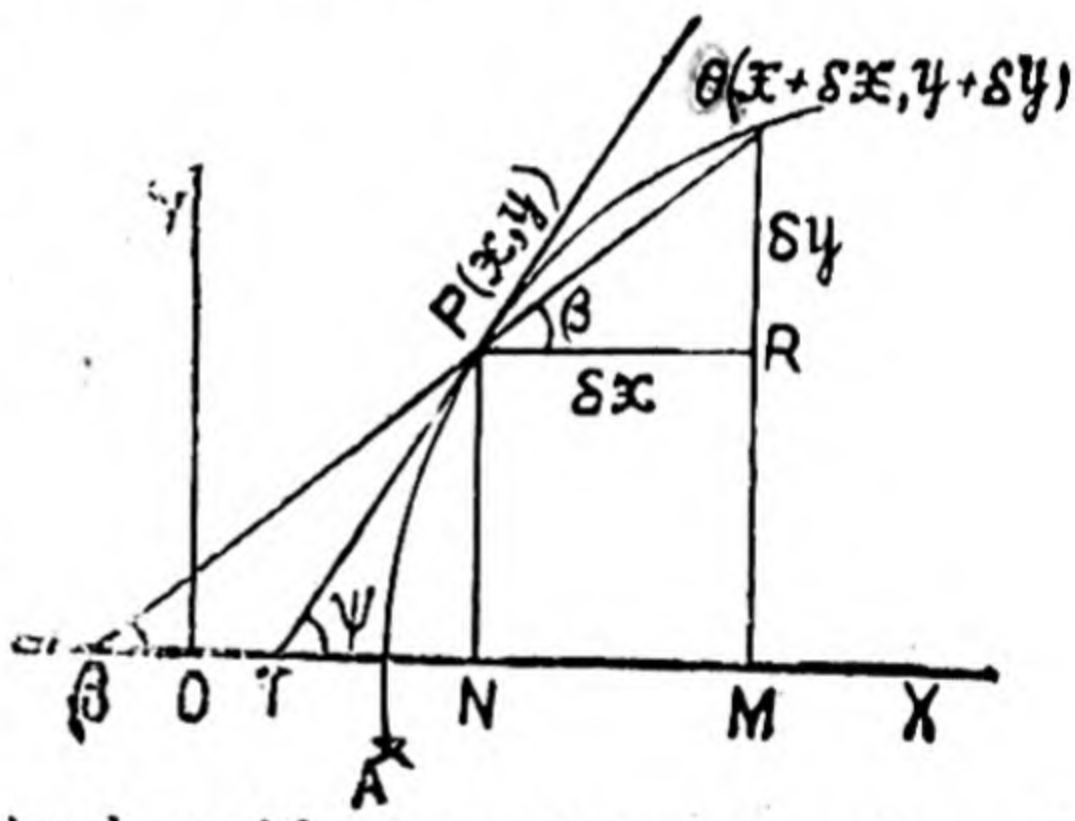
This inequality remains true if the number of the chords constituting the broken line increase indefinitely and their lengths diminish indefinitely in consequence. We **define** "the length of an arc PQ of a curve as the limit to which the enclosed polygon (having P and Q as end points) tends as the *number of chords* si indefinitely increased and conse-



quently the *length* of each chord is indefinitely diminished." This definition of the length of an arc *assumes* that this limit exists and the limit is the same in whatever manner the number of chords is made to increase. Compare the definition of the *circumference* of a circle as the limit of perimeter of an inscribed regular polygon and a similar definition of the *area* of a circle with this.

19. Derivative of arc : Cartesian Co-ordinates.

Let P (x, y) and Q $(x+\delta x, y+\delta y)$ be two neighbouring points of a curve. Let s be the length of the arc AP measured from a fixed point A on it. Evidently s is a function of x or y and we can, therefore, find the derivative of s with respect to x or y. Since arc PQ is the increment of s due to the increment δx of x, we put arc PQ $=\delta s$. Let ψ be the angle the tangent at (x, y) makes with the positive direction of the axis of x.



To prove that

(i) $\dfrac{ds}{dx}=\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

(ii) $\dfrac{ds}{dy}=\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}$.

(iii) $\dfrac{dx}{ds}=\cos\psi$ and $\dfrac{dy}{ds}=\sin\psi$.

Proof. (*i*) Let arc PQ $=\delta s$, chord PQ $=\delta c$.

From $\triangle$PQR, we have

$\delta c=$chord PQ $=\sqrt{(\delta x)^2+(\delta y)^2}$.

$\therefore \dfrac{\delta c}{\delta x}=\sqrt{1+\left(\dfrac{\delta y}{\delta x}\right)^2}$

$$\frac{\delta s}{\delta x}=\frac{\delta s}{\delta c}\cdot\frac{\delta c}{\delta x}=\frac{\delta s}{\delta c}\cdot\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2} \qquad (A)$$

When Q tends to P, $\delta x\to 0$ and we have from (A)

$$\operatorname*{Lt}_{\delta x\to 0}\frac{\delta s}{\delta x}=\operatorname*{Lt}_{\delta x\to 0}\frac{\delta s}{\delta c}\cdot\operatorname*{Lt}_{\delta x\to 0}\left[\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}\right] \qquad (B)$$

But $\operatorname*{Lt}\limits_{\delta c}^{\delta s}=1$

and $$\operatorname*{Lt}_{\delta x\to 0}\sqrt{1+\left(\frac{\delta y}{\delta x}\right)^2}=\sqrt{1+\left(\operatorname*{Lt}\frac{\delta y}{\delta x}\right)^2}$$

$$=\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Hence (B) leads to the formula :

$$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dx}\right)^2}.$$

Note. It may be observed that we must take Q so near to P that arc PQ may be everywhere concave towards its chord.

Cor. If the equation of the curve is given in parametric form,

$x=\phi(t)$, $y=\psi(t)$, then $\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}$ and the formula

(*i*) becomes $$\frac{ds}{dx}=\sqrt{1+\left(\frac{dy}{dt}\Big/\frac{dx}{dt}\right)^2}$$

or $$\frac{ds}{dx}\times\frac{dx}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}$$

i.e., $$\mathbf{\frac{ds}{dt}=\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}}$$

(*ii*) We have

$$\delta c=\sqrt{(\delta x)^2+(\delta y)^2} \quad\therefore\quad \frac{\delta c}{\delta y}=\sqrt{1+\left(\frac{\delta x}{\delta y}\right)^2}$$

Hence

$$\frac{\delta s}{\delta y}=\frac{\delta s}{\delta c}\cdot\frac{\delta c}{\delta y}=\frac{\delta s}{\delta c}\sqrt{1+\left(\frac{\delta x}{\delta y}\right)^2}$$

$$\therefore\quad \text{Lt}\ \frac{\delta s}{\delta y}=\text{Lt}\ \frac{\delta s}{\delta c}\cdot\sqrt{1+\left(\text{Lt}\ \frac{\delta x}{\delta y}\right)^2}$$

$$\therefore\qquad \frac{ds}{dy}=\sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

(*iii*) and (*iv*) Let β be the angle the secant PQ makes with the axis of x.

From $\triangle$PQR we get

$$\cos\beta=\frac{\delta x}{\delta c}=\frac{\delta x}{\delta s}\cdot\frac{\delta s}{\delta c},\ \sin\beta=\frac{\delta y}{\delta c}=\frac{\delta y}{\delta s}\cdot\frac{\delta s}{\delta c}.$$

When Q tends to coincide with P along the arc $\delta x\to 0$ and $\delta s\to 0$, and $\beta\to\psi$.

Therefore proceeding to the limit, we get

$$\text{Lt}\cos\beta=\text{Lt}\ \frac{\delta x}{\delta s}\times\text{Lt}\ \frac{\delta s}{\delta c},$$

$$\text{and Lt}\ \sin\beta=\text{Lt}\ \frac{\delta y}{\delta s}\times\text{Lt}\ \frac{\delta s}{\delta c}$$

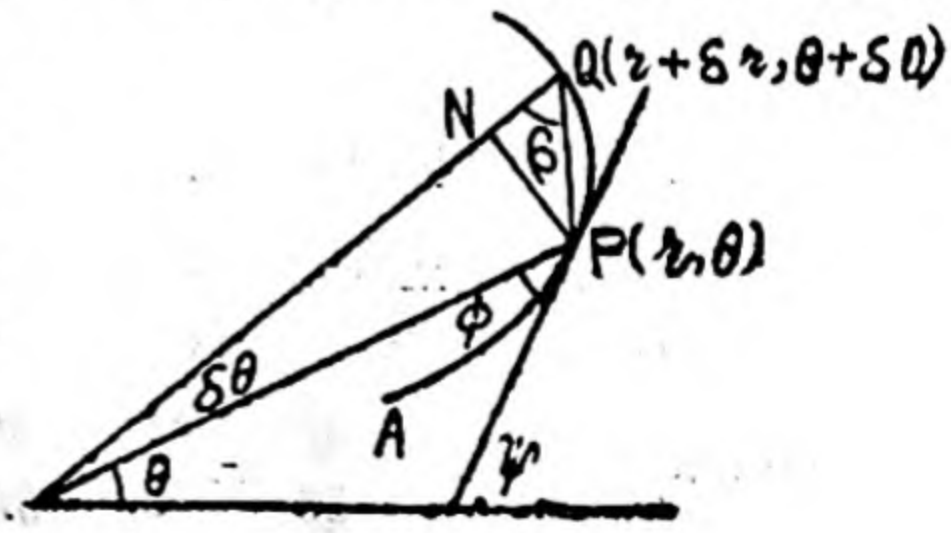
Hence $\cos\psi=\dfrac{dx}{ds}$ and $\sin\psi=\dfrac{dy}{ds}$.

From the last two relations, we deduce

$$1=\left(\frac{dx}{ds}\right)^2+\left(\frac{dy}{ds}\right)^2.$$

20. Derivative of arc : Polar co-ordinates.

Let the co-ordinates of P and its neighbouring point Q be $(r,\ \theta)$ and $(r+\delta r,\ \theta+\delta\theta)$. Let s be the length of the arc AP measured from a fixed point A on it. Then s is a function of r or θ and we can derive s with respect to r or θ. We have, as before arc PQ$=\delta s$; and chord PQ$=\delta c$. Let ϕ be the



angle between the radius vector and the tangent at the point P.

To prove

(*i*) $\dfrac{ds}{d\theta}=\sqrt{r^2+\left(\dfrac{dr}{d\theta}\right)^2}$ (*ii*) $\dfrac{ds}{dr}=\sqrt{1+r^2\left(\dfrac{d\theta}{dr}\right)^2}$

(*iii*) $\cos\ \phi=\dfrac{\mathbf{dr}}{\mathbf{ds}}$ (*iv*) $\sin\phi=\dfrac{\mathbf{rd\theta}}{\mathbf{ds}}$.

First of all observe that

$PN=r\sin\delta\theta$; $NQ=r+\delta r-r\cos\delta\theta$.

Proof. (*i*) We have, from $\triangle PNQ$,

$$\delta c=\sqrt{PN^2+NQ^2}=\sqrt{r^2\sin^2\delta\theta+(r+\delta r-r\cos\delta\theta)^2}$$

Thus $\delta c=\sqrt{r^2\sin^2\delta\theta+\{\delta r+r(1-\cos\delta\theta)\}^2}$ .....(1)

$$\therefore\quad \frac{\delta c}{\delta\theta}=\sqrt{r^2.\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2+\left\{\frac{\delta r}{\delta\theta}+r.\frac{1-\cos\delta\theta}{\delta\theta}\right\}^2}$$

Again $\dfrac{\delta s}{\delta\theta}=\dfrac{\delta s}{\delta c}\cdot\dfrac{\delta c}{\delta\theta}$

$$=\frac{\delta s}{\delta c}\sqrt{r^2\left(\frac{\sin\delta\theta}{\delta\theta}\right)^2+\left(\frac{\delta r}{\delta\theta}+r\frac{1-\cos\delta\theta}{\delta\theta}\right)^2} \qquad \ldots\ldots(2)$$

Now when Q tends to coincidence with P, $\delta\theta\to0$, $\delta s\to0$.

Equation (2) on proceeding to the limit leads to the formula :

$$\frac{ds}{d\theta}=\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

since $\underset{\delta s=0}{\text{Lt}}\ \dfrac{\delta s}{\delta c}=1$, $\underset{\delta\theta=0}{\text{Lt}}\ \dfrac{\sin\delta\theta}{\delta\theta}=1$

and $\underset{\delta\theta=0}{\text{Lt}}\ \dfrac{1-\cos\delta\theta}{\delta\theta}=0.$

(*ii*) From equation (1), we have

$$\frac{\delta c}{\delta r}=\sqrt{r^2.\left(\frac{\sin\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta r}\right)^2+\left\{1+r.\frac{1-\cos\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta r}\right\}^2}$$

Hence $\dfrac{\delta s}{\delta r}=\dfrac{\delta s}{\delta c}\cdot\dfrac{\delta c}{\delta r}$

$$=\frac{\delta s}{\delta c}\times\sqrt{r^2\left(\frac{\sin\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta r}\right)^2+\left\{1+r\frac{1-\cos\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta r}\right\}^2}\quad\ldots\ldots(3)$$

Hence, proceeding to the limit in (3) we get

$$\frac{ds}{dr}=\sqrt{r^2\left(\frac{d\theta}{dr}\right)^2+1}.$$

(*iii*) Let $\angle$PQN be denoted by β.

Then from $\triangle$PQN, we have

$$\cos\beta=\frac{\text{NQ}}{\text{PQ}}=\frac{r+\delta r-r\cos\delta\theta}{\delta c}$$

$$=\frac{\delta r+r(1-\cos\delta\theta)}{\delta s}\cdot\frac{\delta s}{\delta c}$$

$$=\left\{\frac{\delta r}{\delta s}+\frac{r(1-\cos\delta\theta)}{\delta\theta}\frac{\delta\theta}{\delta s}\right\}\cdot\frac{\delta s}{\delta c}.$$

In the limit when Q tends to coincidence with P, $\delta s\to 0$ $\delta\theta\to 0$, and $\beta\to\phi$. Hence proceeding to the limit, we have

$$\cos\phi=\frac{dr}{ds}.$$

(*iv*) We have

$$\sin\beta=\frac{\text{PN}}{\text{PQ}}=\frac{r\sin\delta\theta}{\delta c}=\frac{r\sin\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta s}\cdot\frac{\delta s}{\delta c}.$$

$\therefore$ Lt $\sin\beta=r.$ Lt $\frac{\sin\delta\theta}{\delta\theta}$. Lt $\frac{\delta\theta}{\delta s}$. Lt $\frac{\delta s}{\delta c}$.

Hence $\sin\phi=r\frac{d\theta}{ds}$.

21. For polar co-ordinates, we have in all cases:

$$x=r\cos\theta,\ y=r\sin\theta$$

$\therefore$ $dx=\cos\theta\,dr-r\sin\theta\,d\theta$

$$dy=\sin\theta\,dr+r\cos\theta\,d\theta$$

$\therefore$ $(ds)^2=(dx)^2+(dy)^2=(dr)^2+(rd\theta)^2$

Hence $$\left(\frac{ds}{d\theta}\right)^2=\left(\frac{dr}{d\theta}\right)^2+r^2.$$

$$\left(\frac{ds}{dr}\right)^2=1+\left(\frac{rd\theta}{dr}\right)^2.$$

Since $\tan\phi=\frac{rd\theta}{dr}$, it follows that

$$\cos\phi=\frac{dr}{ds},\ \sin\phi=\frac{rd\theta}{ds}.$$

EXAMPLES XXIX

1. For the curve $y^2=4ax$, prove that $\frac{ds}{dx}=\sqrt{\frac{a+x}{x}}$.

2. For the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$, show that

$$\left(\frac{ds}{dy}\right)^3=\left(\frac{a}{y}\right)^{\frac{2}{3}}.$$

3. In the curve $y=c\cos h\frac{x}{c}$, show that $\frac{ds}{dx}=\frac{y}{c}$.

4. In the curve $y^2(2a-x)=x^3$, show that

$$\frac{ds}{dx}=\frac{a}{2a-x}\left(\frac{8a-3x}{2a-x}\right)^{\frac{1}{2}}.$$

5. For the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, prove that

$$\frac{ds}{d\phi}=a\sqrt{1-c^2\sin^2\phi},\ \text{where } x=a\sin\phi.$$

6. For the curve $x=a(1+\cos\theta)$, $y=a(\theta+\sin\theta)$, show that

$$\frac{ds}{dx}=\operatorname{cosec}\frac{\theta}{2}.$$

7. Show that in the curve

$$x=a(\theta+\sin\theta).\qquad y=a(1-\cos\theta).$$

$$\frac{ds}{d\theta}=2a\cos\frac{\theta}{2},\ \frac{ds}{dy}=\sqrt{\frac{2a}{y}}.$$

8. Show that for the curve $y^2=c^2+s^2$, $\frac{dy}{dx}=\frac{\sqrt{y^2-c^2}}{c}$.

9. Show that for the curve for which $2s=y^2$,

$$\frac{dy}{dx}=\frac{1}{\sqrt{y^2-1}}.$$

10. For the curve, $r^2=a^2\cos 2\theta$, show that

$$\frac{ds}{d\theta}=\frac{a^2}{r}.$$

11. In the curve $r=a\theta$, show that $\frac{ds}{d\theta}=\sqrt{\frac{r^2+a^2}{r}}$.

12. In the curve $r=ae^{\theta\cot\alpha}$ show that $\frac{dr}{ds}=\cos\alpha$.

13. In any curve prove that

(i) $p=r^2\frac{d\theta}{ds}$ (ii) $\sqrt{r^2-p^2}=r\frac{dr}{ds}$.

14. In the curve $r^m=a^m\cos m\theta$, prove that

$$\frac{ds}{d\theta}=a(\sec m\theta)^{\frac{m-1}{m}}.$$

15. For the curve $r^n=a^n\cos n\theta$ show that

$$a^{2n}\frac{d^2r}{ds^2}+nr^{2n-1}=0.$$

16. Prove that the normal at any point (r, θ) to the curve, $r^n=a^n\cos n\theta$ makes an angle $(n+1)\theta$ with the initial line.

CHAPTER IX

ASYMPTOTES OF PLANE CURVES

1. Asymptotes to Algebraic Plane Curve. Before taking up the modern definition of an asymptote which is applicable to all plane curves we shall first deal with algebraic plane curves and give a tentative definition of an asymptote.

An asymptote of a plane algebraic curve is a straight line (not wholly at infinity) which cuts the curve in two points at infinity ; but if in any particular direction there is an infinity of such lines, then out of these particular lines only those which cut the curve in three points at infinity are called asymptotes ; similarly if in a particular direction there is an infinity of lines cutting the curve in three points of infinity then out of these lines only those are called asymptotes which cut the curve in four points at infinity ; and so on.

2. Infinite roots of an equation. Before considering the method of finding asymptotes, we shall first find the condition that two or more roots of the equation

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\ldots\ldots+a_{n-1}x+a_n=0 \qquad \ldots\ldots(1)$$

may be infinite.

Let $x=\frac{1}{y}$. The equation becomes

$$a_0\left(\frac{1}{y}\right)^n+a_1\left(\frac{1}{y}\right)^{n-1}+a_2\left(\frac{1}{y}\right)^{n-2}+\ldots\ldots+$$

$$a_{n-2}\left(\frac{1}{y}\right)^2+a_{n-1}\left(\frac{1}{y}\right)+a_n=0.$$

or $$a_ny^n+a_{n-1}y^{n-1}+a_{n-2}y^{n-1}+\ldots+a_2y^2+a_1y_1+a_0=0 \ldots\ldots(2)$$

It is clear from the relation $x=\frac{1}{y}$ that as y tends to 0, x tends to infinity. Now equation (2) has one root zero if $a_0=0$ and consequently equation (1) has one infinite root if $a_0=0$. Equation (2) has two roots zero if $a_0=0$ and $a_1=0$;

and consequently equation (1) has two roots infinite if $a_0=0$ and $a_1=0$, Similarly equation (1) has three roots infinite if $a_0=0$, $a_1=0$ and $a_2=0$ and so on. Hence we get the following rule : *An equation of nth degree has one infinite root if the co-efficient of x^n is zero ; it has two infinite roots if the co-efficients of x^n and x^{n-1} are both zero ; it has three infinite roots if the co-efficients of x^n, x^{n-1}* and x^{n-2} *are separately zero ; and so on.*

3. Method of finding asymptotes. Now we shall give a method of finding oblique asymptotes (that is asymptotes not parallel to the axes of co-ordinates) of an algebraic curve $f(x, y)=0$ (1) where $f(x, y)$ is a rational algebraic equation of the nth degree in x and y.

Let $y=mx+c$ (2) be an asymptote. Solve these two equations simultaneously for their points of intersection. For this purpose we substitute $mx+c$ for y in the equation to the curve and we get an equation of the nth degree in x giving us the abscissæ of the points of intersection of the curve $f(x, y)=0$ and the line $y=mx+c$. Let this equation in x be

$$b_0x^n+b_1x^{n-1}+b_2x^{n-2}+\ldots : \ +b_n=0.$$

As $y=mx+c$ is an asymptotes, this equation must have two infinite roots and consequently

$$b_0=0 \text{ and } b_1=0. \qquad (3)$$

Solving equations (3) simultaneously for m and c and substituting the values of m and c so obtained in (2) we get, in general, all the asymptotes of the curve. We shall, however, deal with real asymptotes only.

Ex. Find the asymptotes of the curve

$$x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0.$$

Let $y=mx+c$ be an asymptote.

Putting $y=mx+c$ in the above equation we get

$$x^3+2x^2(mx+c)-x(mx+c)^2-2(mx+c)^3+4(mx+c)^2$$
$$+2x(mx+c)+(mx+c)-1=0,$$

or $(1+2m-m^2-2m^3)x^3+(2c-2mc-6m^2c+4m^2+2m)x^2$
$+(-c^2-6c^2m+8cm+2c+m)x-2c^3+4c^2+c-1=0.$

Two roots are infinite, therefore

$$1+2m-m^2-2m^3=0 \qquad \ldots(1)$$

and $$2c-2mc-6m^2c++m^2+2m=0. \qquad \ldots(2)$$

Solving equation (1) for m, we get $m=1$, -1 and $-\frac{1}{2}$.

Making use of (2) for finding values of (c) we get

$c=1$ when $m=\ \ 1$
$c=1$ when $m=-1$
and $c=0$ when $m=-\frac{1}{2}$.

Hence the asymptotes are $y=x+1$, $y=-x+1$ and $y=-\frac{1}{2}x$.

4. Simplification with the help of Taylor's Theorem. The above method for finding asymptotes is sometimes very laborious. But with the help of Taylor's Theorem its working can be simplified.

Let the equation of any curve of the nth degree in homogeneous sets of terms be

$$\begin{aligned}&(a_0x^n+a_1x^{n-1}y+a_2x^{n-2}y^2+\ldots\ldots+a_{n-1}xy^{n-1}+a_ny^n)\\&\quad+(b_1x^{n-1}+b_2x^{n-2}y+\ldots\ldots+b_{n-1}xy^{n-2}+b_ny^{n-1})\\&\quad+(c_2x^{n-2}+c_3x^{n-3}y+\ldots\ldots+c_ny^{n-2})\\&\quad+\ldots\ldots\ldots \qquad =0\end{aligned}$$

$$\text{or}\quad x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots\ldots=0 \qquad (1)$$

where $\phi r\left(\frac{y}{x}\right)$ is an expression of the rth degree in $\frac{y}{x}$.

Let $y=mx+c$ be an asymptote to this curve.

In order to find where this curve is cut by the straight line

$$y=mx+c \qquad (2)$$

we substitute $m+\frac{c}{x}$ for $\frac{y}{x}$ in (1) and the resulting equation is

$$\begin{aligned}&x^n\phi_n\left(m+\frac{c}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{c}{x}\right)\\&\quad+x^{n-2}\phi_{n-2}\left(m+\frac{c}{x}\right)+ \qquad \ldots(3)\end{aligned}$$

which gives the abscissae of the points of intersection.

Expand $\phi_n\left(m+\frac{c}{x}\right)$, $\phi_{n-1}\left(m+\frac{c}{x}\right)$ and so on with the help of Taylor's Theorem this equation becomes

$$x^n\left\{\phi_n(m)+\frac{c}{x}\phi'_n(m)+\frac{c^2}{x^2}\frac{1}{2}\phi_n''(m)\ldots\ldots\right\}$$

$$+x^{n-1}\left\{\phi_{n-1}(m)+\frac{c}{x}\phi'_{n-1}(m)+\frac{c^2}{x^2}\frac{1}{2}\phi''_{n-1}(m)+\ldots\ldots\right\}$$

$$+x^{n-2}\left\{\phi_{n-2}(m)+\frac{c}{x}\phi'_{n-2}(m)+\frac{c^2}{x^2}\frac{1}{\underline{2}}\phi''_{n-2}(m)\ldots\ldots\right\}$$

$$+\ldots\ldots \qquad =0$$

or

$$x^n\phi_n(m)+x^{n-1}\{c\phi'_n(m)+\phi_{n-1}(m)\}$$

$$+x^{n-2}\left\{\frac{c^2}{\underline{2}}\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)\right\} \qquad ..(4)$$

$$+\ldots\ldots=0$$

This is an equation of the nth degree in x showing that a straight line cuts a curve of the nth degree in n points, in general, real or imaginary. As the given line is an asymptote, therefore this equation has two infinite roots. Consequently we have

$$\phi_n(m)=0 \qquad ..(5)$$

$$c\,\phi'_n(m)+\phi_{n-1}(m)=0. \qquad ...(6)$$

If the roots of (5) (which is an equation of nth degree) be $m_1, m_2\ldots\ldots m_n$, then the corresponding values of c will in general be given by $c_1=-\frac{\phi_{n-1}(m_1)}{\phi'_n(m)}$, $c_2=-\frac{\phi_{n-1}(m)}{\phi'_n(m_2)}$.....and consequently the n asymptotes are

$$y=m_1x+c_1,\ y=m_2x+c_2\ldots\ldots y=m_nx+c_n.$$

It may be observed that $\phi_n(m)$ is formed by dividing the terms of the nth degree by x^n and then replacing $\frac{y}{x}$ by m; in practice $\phi_n(m)$ can also be formed from the highest terms by putting $x=1$ and $y=m$.

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Note 1. Equation (4), which gives the directions of the asymptotes has in general n roots, real or imaginary.

Equation (6) is of the first degree in c, so that corresponding to any given value of m, there is only one value of c, we thus deduce that **a curve of the nth degree has n asymptotes, real or imaginary**. If the line at infinity touches the curve, the number of asymptotes will be less than n provided it is not counted as an asymptote.

Note 2. The asymptotes to a curve of the nth degree depend, in general, only on the terms of the nth and the $(n-1)$th degrees in the equation. Hence all curves having the same terms of the two highest degrees have generally the same asymptotes.

Note 3. Taking for granted the following facts from the theory of equations, namely,

(*i*) *that an odd degree* equation has at least one real root ; and

(*ii*) that imaginary roots in an algebraic equation must occur in pairs,

we easily deduce :—

(*a*) *An odd degree curve* must have at least one real asymptote (it must have odd number of real asymptotes) since $\phi_n(m)=0$, in such a case, must be an odd degree equation and must, therefore, have at least one real root. *An odd degree curve*, therefore, must stretch to infinity (having an asymptote) and *cannot be closed curve.*

(*b*) *An even degree curve* has either no real asymptotes or an even number of asymptotes. In this case if the roots of $\phi_n(m)=0$ are all imaginary, the curve would be a closed curve.

Thus a cubic curve cannot be a closed curve having either one or three real asymptotes. A conic has either two real asymptotes (as in the case of a hyperbola) or none (as in the case of an ellipse). An even degree curve may be a closed curve.

Note 4. The highest degree terms in the equation of the algebraic curve are *homogeneous* and of the nth degree. Hence, on equating to zero these terms of the highest

degree, we obtain n straight lines passing through the origin whose joint equation is $\phi_n\left(\frac{y}{x}\right)=0.$ (1)

These *lines are evidently parallel to the asymptotes* whose directions are given $\phi_n(m)=0.$ (2)

Thus we notice that *the n lines through the origin obtained by equating to zero the highest degree terms of the equation of a curve are parallel to the n asymptotes.* They are not the asymptotes except in special cases which will be indicated later.

Ex. Find the asymptotes of

$$x^3+2x^2y-xy^2-2y^3+4y^2+2xy+y-1=0.$$

Here $\phi_n(m)=1+2m-m^2-2m^3=0$ from which we get $m=1, -1, -\frac{1}{2}$.

Also $\phi_n'(m)=2-2m-6m^2$ and $\phi_{n-1}(m)=4m^2+2m,$

and consequently c giving equation of $c\phi'_n(m)+\phi_{n-1}(m)=0,$ becomes $c(2-2m-6m^2)+4m^2+2m=0.$

or $$c=-\frac{2m^2+m}{1-m-3m^2}$$

For $m=1, -1, -\frac{1}{2},$

$c=1, 1, 0.$

Hence the asymptotes are

$$y=x+1$$
$$y=-x+1$$
$$y=-\tfrac{1}{2}x.$$

EXAMPLES XXX

Find the asymptotes of the following curves :—

1. $x^2-3xy^2+3x-y+4=0.$
2. $y^3-6xy^2+11x^2y-6x^3+2x+y=0.$
3. $y^3-x^2y+2y^2+4y+x=0.$
4. $x^3+2x^2y-xy^2-2y^3+xy-y^2+x=0.$
5. $x^3-y^3=3xy.$
6. $2x^3-x^2y-2xy^2+y^3+2x^2+xy-y^2+3x+y+5=0.$
7. $y^3-x^2y+2y^2+4y+1=0.$

Find the asymptotes of the following curves and verify in each case that the asymptotes are parallel to the lines obtained by equating to zero the linear factors of the terms *of the highest degree* :

8. $y^3=6x^2+x^3$. **9.** $x^3+y^3=a^3$. **10.** $y^4-x^4-2bx^2y=0$.

11. $y^3=2ax^2-x^3$. **12.** $y^2=4x^2+2x+6$. **13.** $y^3=x^2(2a-x)$.

Find the asymptotes of the following curves :—

14. $x^3-6x^2y+11xy^2-6y^3+x+y=0$. **15.** $y^3-x^3=3x^2$.

16. $x^3+2yx^2-xy^2-2y^3+5x^2+4xy-3y^3+4y+3x+5=0$.

17. $x^3-4y^2x-yx^2+4y^3+4xy-4y^2-5=0$.

18. $(x+y-1)(x+3y-1)(3x-y+2)+3x+4y+5=0$.

19. $(x^2-y^2)(2x+3y)(3x+2y)+2x^2+3y^2-6=0$.

5. Asymptotes parallel to the axes.

Let the equation of the curve as given in Art. 4 be arranged in descending powers of x so that it takes the form

$$a_0x^n+(a_1y+b_1)x^{n-1}+(a_1y^2+b_2y+c_2)x^{n-2}+\ldots\ldots=0. \quad \ldots(1)$$

If a_0 is zero, then $a_1y+b_1=0$ is an asymptote to the curve parallel to the x-axis, because for points of intersection of $a_1y+b_1=0$ and (1) we put $y=-\frac{b_1}{a_1}$ in (1), it reduces to an equation in which the co-efficients of the two highest powers of x are zero and therefore two of its roots are infinite.

Again, if a_0, a_1, b_1 are all zero, then the pair of lines

$$a_2y^2+b_2y+c_2=0,$$

are asymptotes (real or imaginary) because these lines cut the curve in three points at infinity and so on.

Similarly if $a_n=0$, then $a_{n-1}x+b_n=0$ is an asymptote parallel to the y-axis and so on.

Thus we deduce the *following rules* for finding asymptotes parallel to the axes :—

Rulo 1. (*i*) If the term involving x^n be absent in the equation of a curve of the nth degree, the co-efficient of x^{n-1} (this co-efficient being linear in y) equated to zero gives the equation of an asymptote parallel to the axis of x.

(*ii*) If the terms involving x^n and x^{n-1} be both absent, then the co-efficient of x^{n-2} (this co-efficient is usually a quadratic in y) equated to zero gives two asymptotes parallel to the axis of x; and so on.

Rule 2. (*i*) If the term containing y^n be absent, the co-efficient of y^{n-1} (co-efficient being linear in x) equated to zero gives an asymptote parallel to the y-axis. (See Art. 13.)

(*ii*) If the terms involving y^n and y^{n-1} be both absent, then the co-efficient of y^{n-2} (this co-efficient being usually a quadratic in x) equated to zero gives two asymptotes parallel to the y-axis; and so on.

Note. Observe that if the co-efficients do not involve y as in Rule 1 (or x in Rule 2), they cannot be zero and there will be no corresponding asymptote.

Ex. 1. Find the asymptotes parallel to co-ordinate axes in the curve $x^2y^2=a^2(x^2+y^2)$

(*i*) This is a *fourth degree curve*, co-efficients of y^1 and y^3 both wanting $\therefore$ coefficients of y^2 equated to zero will give us two asymptotes. Arranging the equation of the curve according to descending powers of y, we have

$$(x^2-a^2)y^2-a^2x^2=0$$

$\therefore$ $x^2-a^2=0$, *i.e.*, $x=a$ and $x=-a$ are the two asymptotes parallel to the axis of y.

(*ii*) Again, the co-efficients of x^1 and x^3 are both wanting from the fourth degree curve. Arranging according to descending powers of x, we have

$$(y^2-a^2)x^2-a^2y^2=0.$$

$\therefore$ the co-efficient of x^2 equated to zero will give us two asymptotes.

Ex. 2. Find the asymptotes parallel to co ordinate axes in the curve $y^2a=y^2x+x^3$.

(*i*) This is a *third degree* curve and the *co-efficient of* y^3 *wanting* $\therefore$ co-efficient of y^2 equated to zero will give us an asymptote. Hence $x=a$ is an asymptote parallel to y-axis.

(*ii*) since the term in x^3 is present, therefore there is no asymptote parallel to x-axis.

Ex. 3. Examine, for asymptotes parallel to axes, the curve $y=(y-1)(x-3)/(x-2)(x-4)$.

(*i*) This is a *third degree* curve and the terms in y^3 and y^2 are wanting. Therefore, the co-efficient of y equated to zero will give two asymptotes $\therefore$ $x-2=0$ and $x-4=0$ are two asymptotes $\parallel$ to y-axis.

(*ii*) Again the term involving x^3 is wanting, $\therefore$ the co-efficient of x^2 (which involves y) equated to zero will give an asymptote. Hence $y-1=0$ is the required asymptote parallel to x-axis.

Ex 4. Find the asymptotes parallel to the axes of the curve $y^2=(x-1)(x-3)/(x-2)(x-4)$.

This is a fourth degree curve; it can be easily shown exactly as in Ex. 3 that $x=2$, $x=4$, and $y=\pm 1$ are the asymptotes.

Note. In examples of this type where a power of y stands singly and alone on the left, and the right-hand side is a fraction involving only x, and powers of x, all the *linear factors of the denominator equated to zero are asymptotes*, (provided none of these factors are factors of the numerator or the differential co-efficient of the numerator).

EXAMPLES XXXI

Find the asymptotes parallel to the axes :—

1. $xy^3+x^3y=a^4$.
2. $x^2y^2=a^2(x^2+2)$.
3. $x^2y^2=a^2(x^2-y^2)$.
4. $x^4+x^2y^2+a^2x^2-a^2y^2=0$.
5. $ax^3+x^3y-ay^3=0$.
6. $x^3+y^3=c^3$.
7. $x^3-x^2y+y+x=0$.
8. $x^4-y^4+a^2yx=0$.
9. $xy^2+ay^2=x$.
10. $xy^2-x^2y+x+y=0$.
11. $(x^3+a^3)y=bx^3$.
12. $x^2y^3+x^3y^2=x^3+y^3$.

6. Particular cases. We have already seen in Art. 4 that equations (5) and (6) in general determine m and c for the asymptotes. However particular cases sometimes arise which need further investigation.

I. Suppose that two of the roots of the equation $\phi_n(m)=0$ are equal, say $m_1=m_2$; then $\phi'_n(m_1)=0$. In this case if $\phi_{n-1}(m_1)$ is not equal to zero, then the value of c corresponding to m_1 obtained from equation (6), namely

from the equation $c\phi'_n(m_1)+\phi_{n-1}(m_1)=0$ is infinite and consequently the line $y=m_1x+c_1$ is wholly at infinity and is therefore not an asymptote.

II. However if $\phi_n(m)=0$ has a pair of equal roots, say each equal to m_1, we have $\phi'_n(m_1)=0$; and if $\phi_{n-1}(m_1)$ is also zero, then the value of c corresponding to m_1 cannot be determined from the equation $c\phi'_n(m)+\phi_{n-1}(m)=0$. It follows from equation (4) of the previous article that whatever c may be the co-efficients of x^n and x^{n-1} are zero when $m=m_1$ and consequently every line parallel to $y=m_1x$ cuts the curve in two points at infinity. According to the definition of an asymptote out of these only those lines are called asymptote which cut the curve in three points at infinity. We therefore select c such that the co-efficient of x^{n-2} in equation (4) is zero, that is to say, we select c such that

$$\frac{c^2}{\lfloor 2}\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0,$$

which gives two values of c corresponding to $m=m_1$. Let these values be c_1 and c_2. We thus get two parallel asymptotes

$$y=m_1x+c_1 \text{ and } y=m_1x+c_2.$$

Note I. This case occurs when the highest degree terms in the equation of the curve have a factor of the form $(y-mx)^2$ and the terms of the next lower degree have the factor $y-mx$.

Note II. If the co-efficient of x^{n-2} also vanishes identically independent of c, that is to say, if

$$\phi''_n(m_1)=0,\ \phi'_{n-1}(m_1)=0 \text{ and } \phi_{n-2}(m_1)=0,$$

we choose c from the cubic equation obtained by equating to zero the co-efficient of x^{n-3}; which means from the equation

$$\frac{c^3}{\lfloor 3}\phi'''_n(m_1)+\frac{c^2}{\lfloor 2}\phi''_{n-1}(m_1)+c\phi'_{n-2}(m_1)+\phi_{n-3}(m_1)=0.$$

We thus get three parallel asymptotes. This occurs when $\phi_n(m)=0$ has three roots each equal to m_1. And so on.

Find the asymptotes of :

Ex. 1. $x^3+4x^2y+5xy^2+2y^3+2x^2+4xy+2y^2-x-9y+1=0.$

Here $\phi_n(m)=0$ gives

$$1+4m+5m^2+2m^3=0$$

or $$(m+1)^2(2m+1)=0.$$

Therefore $$m=-1,\ -1,\ -\tfrac{1}{2}.$$

c giving equation is $c\phi'_n(m)+\phi_{n-1}(m)=0$

i.e., $$c(4+10m+6m^2)+(2+4m+2m^2)=0.$$

For $m=-\frac{1}{2}$, $c=-1$ and the corresponding asymptote is $y=-\frac{1}{2}x-1.$

or $2y+x+2=0.$

For $m=-1$, this equation is identically satisfied. Therefore we proceed further. Now the c giving equation is

$$\frac{c^2}{2}\phi''_n(m)+c\phi'_{n-1}(m)+\phi_{n-2}(m)=0, \text{ which becomes}$$

$$\frac{c^2}{2}(10+12m)+c(4+4m)+(-1-9m)=0.$$

For $m=-1$, we get

$-c^2+8=0$, therefore $c=\pm 2\sqrt{2}$.

Hence the corresponding asymptotes are

$$y=-x\pm 2\sqrt{2}.$$

Ex. 2. Find the asymptotes of

$$x^4+2x^3y-2xy^3-y^4-x^2+y^2+3x+3y=0.$$

Here $\phi_n(m)=1+2m-2m^3-m^4=0$,

or $(1+m)^3(1-m)=0$,

whence $m=-1,\ -1,\ -1,\ 1.$

c giving equation is

$$c(2-6m^2-4m^3)=0.$$

For $m=1$, this gives $c=0$

and the corresponding asymptote is $y=x.$

For $m=-1$, this becomes an identity and therefore does not give c; we therefore take the next co-efficient and equate it to zero. We get

$$\frac{c^2}{2}(-12m-12m^2)+(-1+m^2)=0.$$

For $m=-1$, this again is 0 whatever c may be.

We therefore proceed further and take the equation

$$\frac{c^3}{\lfloor 3}\phi'''_n(m)+\frac{c^2}{\lfloor 2}\phi''_{n-1}(m)+c\phi'_{n-2}(m)+\phi_{n-3}(m)=0$$

that is

$$\frac{c^3}{\lfloor 3}(-12-24m)+c(2m)+3+4m=0$$

or $$\frac{c^3}{6}(-12+24)-2c+3-3=0$$

or $$2c^3-2c=0,$$

whence $c=0$ and $c=\pm 1$;

therefore the corresponding asymptotes are

$$y=-x,\ y=-x\pm 1$$

or $$y+x=0,\ y+x=\pm 1.$$

EXAMPLES XXXII

Find the asymptotes of the following curves :=

1. $x^3-5x^2y+8xy^2-4y^3-3x^2+9xy-6y^2+2x-2y=1.$
2. $x^3-x^2y-xy^2+y^3-x^2+y^2-1=0.$
3. $(x^2-y^2)^2-2(x^2+y^2)+c-1=0.$
4. $y^3-5xy^2+8x^2y-4x^3-3y^2+9xy-6x^2+2y-2x-1=0.$
5. $y^3+2y^2x-4yx^2-8x^3-4y+8x=1.$
6. $x^3+2x^2y+xy^2-x^2-xy+2=0.$
7. $x^3-2x^2y+xy^2+x^2-xy+2=0.$
8. $yx^3+3y^3x-3x^2y^2-y^4-2y^2+y^3-2=0.$
9. $x^2(x+y)(x-y)^2+ax^3(x-y)-a^2y^3=0.$

7. Miscellaneous Methods. The methods given so far for algebraic curves are complete in themselves and will give all asymptotes. However we propose to give a few more methods which when judiciously applied shorten our work.

I. If the equation of a *curve* can be thrown in the form

$$(ax+by+c)P_{n-1}+E_{n-1}=0 \qquad (I)$$

where P_r and F_r denote rational algebraic expressions which contain r^{th} and lower degree terms but no higher, then,

(*i*) $ax+by=0$ is parallel to an asymptote, since it is a factor of the highest degree terms (nth degree terms) which are evidently contained in $(ax+by)P_{n-1}$.

(*ii*) Now $ax+by=\dfrac{-F_{n-1}}{P_{n-1}}-c$ (*curve*)

$\therefore$ the asymptote $\parallel$ to $ax+by=0$ is

$$ax+by=\text{Lt}\left(-\frac{F_{n-1}}{P_{n-1}}-c\right)$$

i. e., $ax+by+c=\text{Lt}\left(\dfrac{-F_{n-1}}{P_{n-1}}\right)$ when x and y tend to infinity in the ratio $y/x=-\dfrac{a}{b}$.

There is an asymptote parallel to $ax+by+c=0$, if the limit on the right exists.

II. If the second term in equation (1) instead of being F_{n-1} be F_{n-2} which contains only $(n-2)$th and lower degree terms but no higher, then the equation of the curve becomes

$$(ax+by+c)P_{n-1}+F_{n-2}=0. \qquad \text{(II)}$$

In this case, $ax+by+c=0$ *is itself an asymptote* provided P_{n-1} does not contain $(ax+by+c)$ as a factor.

For, $ax+by+c=0$ may be expressed in the form $y=-a/b.x-c/b$ and if for y we *substitute* $-a/b.x-c/b$ in equation (II), the co-efficients of x^n and x^{n-1} are evidently zero.

$\therefore$ $ax+by+c=0$ is an asymptote.

Exceptional Case. If in equation (II) P_{n-1} contains $ax+by+c$ as a factor: $P_{n-1}=(ax+by+c)P_{n-2}$, say, then equation (II) is of the form

$$(ax+by+c)^2P_{n-2}+F_{n-2}=0,$$

and the curve has *two parallel asymptotes*

$$ax+by+c=\pm\sqrt{-\text{Lt}\frac{F_{n-2}}{P_{n-2}}},$$

where x and y tend to ∞ in the ratio $y/x=-a/b$.

Note. The evaluation of the limit on the right-hand side could be effected by putting $y=-a/t$, $x=b/t$ and then making $t\to 0$.

Ex. 1. Find the asymptotes of the curve
$(x+y+1)(x^2+y^2-xy)-3xy+x^2+y^2+2x-3y+5=0.$

The asymptote $\parallel$ to $x+y+1=0$ is given by

$$x+y+1=\operatorname*{Lt}_{y=-x=\infty}\frac{3xy-x^2-y^2-2x+3y-5}{x^2+y^2-xy}$$

$$=\frac{3y/x-1-(y/x)^2-2/x+3y/x^2-5/x^2}{1+(y/x)^2-y/x}$$

$$=\frac{-3-1--1}{1+1+1}=-\frac{5}{3}.$$

$\therefore$ $x+y+8/3=0$, *i. e.*, $3(x+y)+8=0$ is the only real asymptote.

The evaluation of the limit may have been effected by putting $y=-1/t$, $x=1/t$ and then making $t\to 0$.

Note. Observe that the limit would have been the same if we had retained only the terms of the second degree in the numerator and the denominator on the right.

Ex. 2. Find the asymptotes of the curve
$(x+y+1)^2(x^2+y^2-xy)+3xy-7x^2-2y^2-7x+8=0.$

There are two parallel asymptotes and these are given by

$$x+y+1=\sqrt{\operatorname*{Lt}\frac{-3xy+7x^2+2y^2+7x-8}{x^2+^2-xy}}$$

where x and $y\to\infty$ in the ratio $y/x=-1$

$$=\sqrt{\operatorname*{Lt}\frac{-3y/x+7+2y/x)^2}{1+(y/x)^2-y/x}}$$ rejecting the terms $7x-8$ from the numerator).

$=\pm 2.$

$\therefore$ the two parallel asymptotes are $x+y+3=0$ and $x+y-1=0$.

IV. Asymptotes by Inspection. If the equation of a *curve* be

$$F_n+F_{n-1}=0 \qquad \text{.....(1)}$$

then every linear factor of F_n equated to zero will give an asymptote provided that no two asymptotes so obtained are parallel or coincident. Thus, if

$$F_n \equiv (a_1x+b_1y+c_1)(a_2x+b_2y+c_2)\ldots(a_nx+b_ny+c_n),$$

then $a_1x+b_1y+c_1=0$ is an asymptote ; for this may be written as $y=-a_1/b_1.x-c_1/b_1$ and if we substitute $-a_1/b_1.x-c_1/b_1$ for y in (1), the co-efficients of x^n and x^{n-1} are obviously zero.

Similarly, $a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$

.....

and $a_nx+b_ny+c_n=0$ are asymptotes.

Ex. 1. Find the asymptotes of the curve

$$xy(x^2-y^2)+2x^2+2xy+2y^2=1.$$

This is of the form $F_n+F_{n-2}=0(n=4)$.

Since none of the linear factors of F_n are repeated the *four asymptotes* are $x=0$, $y=0$, $x-y=0$, $x+y=0$.

Ex. 2. Find the asymptotes of the curve

$$x^2y-xy^2+xy+y^2+x-y=0.$$

This can be written in the from

$$x^2y-xy^2+xy+y^2-2y+x+y=0$$

i. e., $y(x^2-xy+x+y-2)+x+y=0$

i. e., $y\{(x^2-1)+(x-1)-y(x-1)\}+x+y=0$

i. e., $y(x-1)(x-y+2)+x+y=0$.

This is of the form $F_n+F_{n-2}=0$ $(n=3)$.

Since none of the factors are repeated. the *three asymptotes* are

$$y=0\ ;\ x-1=0\ ;\ x-y+2=0.$$

Ex. 3. Find the asymptotes of the curve

$$(x^2-y^2)^2-4y^2+y=0.$$

The equation is of the form $F_n+F_{n-2}=0(n=4)$.

But the factors of F_n are repeated, therefore, corresponding to each repeated factor, there are two parallel asymptotes

$$(x-y)^2=\frac{4y^2-y}{(x+y)^2}\ ,$$

$\therefore$ the two asymptotes parallel to $x-y=0$ are

$$x-y=\pm\sqrt{\operatorname{Lt}\frac{4y^2-y}{x^2+y^2+2xy}}=\pm\sqrt{\frac{\operatorname{Lt}\,4(y/x)^2-y/x^2}{1+(y/x)^2+2(y/x)}}$$

when x and y tend to infinity in the ratio $y/x=1$.

i. e., $x-y=\pm 1$ are parallel asymptotes in the direction $y-x=0$.

Similary $x+y=\pm 1$ are the two parallel asymptotes in the directlon $x+y=0$.

8. Intersection of a curve with its Asymptotes.

Suppose an algebraic curve of the nth degree has n asymptotes, no two of which are parallel, then

(*i*) $F_n+F_{n-2}=0$ is the *curve*.

(*ii*) $F_n=0$ is the joint equation of the asymptotes.

Hence at the common points of intersection of (*i*) and (*ii*), we must have $F_{n-2}=0$. Hence *the point of intersection of a curve and its asymptotes lie on the locus* $F_{n-2}=0$ which is a curve of the $(n-2)$th degree. Sinee each asymptote meets the curve in $(n-2)$ points, excluding the points at infinity, *the n asymptotes will cut the curve again in* $n(n-2)$ *points, which lie on the locus* $F_{n-2}=0$. In the special cases, of the cubic ($n=3$) and quartic ($n=4$) $F_{n-2}=0$ will become a *straight line* and a *conic* respectively and we easily deduce that

(*i*) *the three points in which the asymptotes of a cubic meet the curve lie on a right line.*

(*ii*) *the eight points in which the four asymptotes of a quartic meets the curve lie on a conic section.*

It may be observed that if the joint equation of all the asymptotes of a curve is $F_n=0$, then the equation to the curve can always be put in the form $F_n+F_{n-2}=0$, where F_{n-2} is an expression of the $(n-2)$th degree and consequently the points of intersection of the curve and its asymptotes lie on the locus $F_{n-2}=0$.

Ex. Show that the asymptotes of the curve

$$(x^2-y^2)(y^2-4x^2)-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0$$

cut the curve again the eight points which lie on a circle.

We shall first find the asymptotes

$$\phi_n(m)=(1-m^2)(m^2-4)=0.$$

Therefore $m=\pm1, \pm2,$

$c\phi'_n(m)+\phi_{n-1}(m)=0$ becomes

$c(10m-4m^3)+(-6+5m+3m^2-2m^3)=0.$

For $m=1, c=0$

$m=-1, c=-1$

$m=2, c=0$

$m=-2, c=-1.$

The four asymptotes are

$$y=x,\ y=-x-1,\ y=2x,\ y=-2x-1.$$

The joint equation of the four asymptotes, therefore, is

$$(y-x)(y+x+1)(y-2x)(y+2x+1)=0$$

or $5x^2y^2-4x^4-y^4-2y^3+3xy^2+5x^2y-6x^3-y^2+3xy-2x=0.$

The equation of the curve can be written as

$$(5x^2y^2-4x^4-y^4-2y^3+3py^2+5x^2y-6x^3-y^2+3xy-x^2)$$
$$+(x^2+y^2-1)=0,$$

which is of the form $F_4+F_2=0.$

Hence the points of intersection [which are $4(4-2)$ or 8, in number] lie on the circle

$$x^2+y^2=1.$$

EXAMPLES XXXIII

Find the asymptotes of the following curves :—

1. $y^2(x^2-a^2)=x.$
2. $y^2(x-a)=x^3+ax^2.$
3. $y^2(x-1)(x-2)=x^2+4.$
4. $xy(x^2-y^2)+a^2y^2+b^2x^2=a^2b^2.$
5. $xy(x^2-y^2)(x^2-9y^2)+2xy(x^2-y^2)+x^2+y^2=1.$
6. $(x+y)^2(x+2y)(x+3y)-2(x^3+y^3)-2(x+2y)(x+y)=0.$
7. $(x+2y)(x^2+xy+y^2)=x^2+y^2+x.$
8. $(x+y)(x^4+y^4)=2(x^4+16).$
9. (*i*) $(2x-3y+1)^2(x+y)-8x+2y-9=0.$

   (*ii*) $(x+y)^2(x+2y+2)=x+9y-2.$

10. $(y-x)^2(x^2+y^2)-10(y-x)y^2+12x^2+2y+x=0.$

11. $x(y-x)^3=x(y-x)+2.$

12. $(x+y)^2(x+2y+2)=x+9y-2.$

Generalise the method in each case you employ.

13. $(x^2-a^2)y^2=x^2(x^2-4a^2).$

14. $(x^2-y^2)(y^2-4x^2)-6x^3+5x^2y+3xy^2-2y^3-x^2+3xy-1=0.$

15. $(x-y+1)(x-y-2)(x+y)=8x-1.$

16. $(x-y)^2(x-2y)(x-3y)-2a(x^3-y^3)-2a^2(x+y)(x-2y)=0.$

17. Show that the four asymptotes of the curve

$$xy(x^2-y^2)+25y^2+9x^2=144$$

cut it again in eight points lying on an ellipse whose eccentricity is $4/5$.

18. Find the asymptotes of the cubic curve

$$x^2y-xy^2+xy+y^2+x-y=0,$$

and show that they cut the curve again in three points which lie on the line $x+y=0$.

19. From the equation of a quartic curve which has

$$x=0,\ y=0,\ y=x \text{ and } y=-x$$

as asymptotes, which passes through the points (a, b) and cuts its asymptotes again in eight points lying on the circle

$$x^2+y^2=a^2.$$

20. Show that the four asymptotes of the curve

$$(x^2-y^2)(y^2-4x^2)+6x^3-5x^2y-3xy^2+2y^3-x^2+3xy=1$$

cut the curve again in points which lie on a circle of unit radius.

21. Show that the equation of a cubic which has the same asymptotes at the cubic curve

$$x^3-6x^2y+11xy^2-6y^3+x+3y+5=0$$

and which passes through the points (0, 0), (1, 0) and (0, 1) is

$$x^3-6x^2y+11xy^2-6y^3-x+6y=0.$$

22. Show that $x=c$ is an asymptote to the curve

$$y(x-c)=f(x) \text{ if } f(c) \text{ and } f''(c) \text{ are infinite.}$$

23. If it is possible to put the equation of a curve in the from $y=mx+c+A/x+B/x^2+\ldots$ m, c, A, B...being constants, show that the line $y=mx+c$ is an asymptote to the curve approaches the asymptote from above or from below according as A is positive or negative.

24. Apply the method of Exercise 21 to find the oblique asymptote of $y(x^2-a^2)=x(x^2+a^2)$ and show that in the first quadrant the curve approaches the asymptote from above.

$$[y/x=(x^2+a^2)/(x^2-a^2)=\frac{1+a^2/x^2}{1-a^2/x^2}=$$

$$(1+a^2/x^2)(1+a^2/x^2+a^4/x^4+\ldots\ldots)$$
$$=1+2a^2/x^2+\ldots\ldots \text{or } y=x+2a^2/x+\ldots.$$

Hence $y=x$ is an asymptote etc., etc.]

25. Show that the oblique asymptotes of the curve $y^2(x-a)=x^2(x+a)$ are $y=x+a$, and $y=-x-a$, and that in the first quadrant the curve approaches the former asymptote from above.

26. Give the asymptotes of a cubic, find its equation, showing that the three points in which the asymptotes meet the curve lie on a st. line.

9. Asymptotes of Plane Curves—General. An asymptote (to an infinite branch) of a curve is a straight line (not at infinity) the distance of which from a point on the curve diminishes without limit as the point moves along the curve to an infinite distance from the origin.

Let $P(x, y)$ be a point on the curve $f(x, y)=0$ and δ, the perpendicular from the point P on a line $y=mx+c$; then, if $\delta\to 0$, as $x\to\infty$ or $x\to\infty$ or both of them tend to ∞, the line $y=mx+c$ is called an asymptote to the curve $f(x, y)=0$.

10. We proceed to determine the values (*finite*) of m and c in order that $y=mx+c$ may become an asymptote to $f(x, y)=0$.

The *length of the perpendicular* (irrespective of sign) from a point $P(x, y)$ on the curve $f(x, y)=0$ to $y=mx+c$ is

$$\delta\equiv\frac{y-mx-c}{\sqrt{1+m^2}}. \qquad (1)$$

$$\therefore \frac{\delta}{x} = \frac{\frac{y}{x} - m - \frac{c}{x}}{\sqrt{1+m^2}}.$$

When $x \to \infty$ or $-\infty$ (according to the sense in which the branch stretches to infinity, $\delta \to 0$ (*definition*)

$$\therefore \frac{\delta}{x} \to 0 \; ; \text{ also } \frac{c}{x} \to 0.$$

(*i*) $\therefore \; m = \text{Lt} \left(\frac{y}{x}\right)$ when x tends to infinity.

Again from (1) when m has been determined, since $\delta \to 0$, we have

(*ii*) $c = \text{Lt}\,(y - mx)$, when x tends to infinity.

We thus deduce the *following method* for determining the equation of an asymptote :—

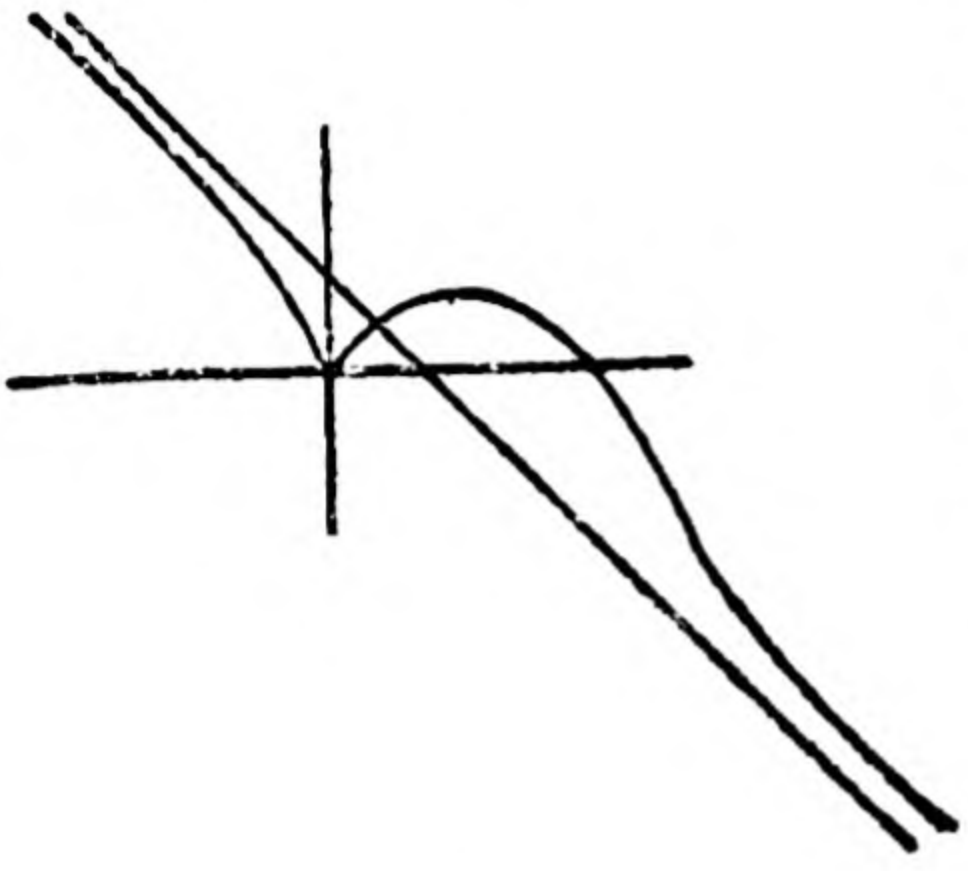
If an infinite branch of a curve possesses any asymptote not parallel to y-axes, $y = mx + c$, the co-efficients m and c are given by the equations :

(1) $\mathbf{m = \text{limit } \frac{y}{x}}$, $\mathbf{c = \text{limit } (y - mx)}$ *where x tends to infinity, the point P (x, y) always moving on the branch.*

Note 1. A similar method determines asymptotes of the form $x = my + c$, *i.e.*, those parallel to y-axis as in this case $m = \text{Lt } x/y$ and $c =$, *at* $(x - my)$ when $y \to \infty$.

Note 2. The method of Indeterminate Forms can be employed for determining "m" and "c" since Lt y/x and Lt $(y - mx)$ are indeterminate being respectively of the forms '∞/∞' and "$\infty - \infty$" when x and y tend to infinity. But these limiting values are more readily found from the equation of the curve without recourse to differentiation as the following examples will show.

Ex. 1. Examine the curve $x^3 + y^3 - 6x^2 = 0$ for asymptotes.



$y=mx+c$ is an asymptote

where $m=\text{Lt}\left(\frac{y}{x}\right)$, $c=\text{Lt}\,(y-mx)$ } as $x\to\infty$

(*i*) To find m, we have

$$x^3+y^3-6x^2=0$$

i.e., $$1+\left(\frac{y}{x}\right)^3-\frac{6}{x}=0.$$

Now when $x\to\infty$,

$$\text{Lt}\,\frac{6}{x}=0 \text{ and}$$

$$\text{Lt}\,\frac{y}{x}=m$$

$\therefore$ $m^3+1=0$, which gives the directions of the asymptotes.

$m=-1$ is the only real root, the other two roots are imaginary. There is, therefore, only one real asymptote.

$\text{Lt}\,\frac{y}{x}$ will be the same whether $x\to+\infty$ or $x\to-\infty$.

(*ii*) Again

$c=\text{Lt}\,(y-mx)=\text{Lt}\,(y+x)$ when x and y tend to ∞.

Now $(y+x)(x^2+y^2-xy)=6x^2$ (*curve*)

$\therefore$ $$y+x=\frac{6x^2}{x^2-xy+y^2}$$

$\therefore$ $$\text{Lt}\,(y+x)=\text{Lt}\,\frac{6x^2}{x^2-xy+y^2},\ x\to\infty$$

$$=\text{Lt}\,\frac{6}{1-\left(\frac{y}{x}\right)+\left(\frac{y^2}{x^2}\right)},\ x\to\infty$$

$$=\frac{6}{1+1+1}, \text{ since } \text{Lt}\,\frac{y}{x}=m=-1$$

$\therefore$ $$c=2.$$

(*iii*) Substituting these values of m and c in $y=mx+c$, we get $y+x=2$ as the equation of the only real asymptote.

Ex. 2. Examine for asymptotes the curve

$$x^2+3xy+2y^2+3x-2y+1=0.$$

$y=mx+c$ is an asymptote,

where $m=\text{Lt}\ \dfrac{y}{x}$, $\quad c=\text{Lt}\ (y-mx)$ $\Big\}$ as x and y tend to ∞.

The asymptotes exist if these limits exist.

(*i*) *To find m*, we have, from the equation of the curve, by dividing by x^2,

$$1+3\,\frac{y}{x}+2\left(\frac{y}{x}\right)^2+\frac{3}{x}\ -2\frac{y}{x^2}+\frac{1}{x^2}=0.$$

In the limit when $x\to\infty$, $y\to\infty$

$\text{Lt}\ \dfrac{y}{x}=m$, we get $1+3m+2m^2=0$

$\left(\because\ \text{Lt}\ \dfrac{3}{x}=0\ ;\ \text{Lt}\ \dfrac{1}{x^2}=0\ ;\ \text{Lt}\ \dfrac{y}{x^2}=0.\ \right)$

i e., $(2m+1)(m+1)=0$

$\therefore\quad m_1=-1$, and $m_2=-\dfrac{1}{2}$.

(*ii*) Taking $m_1=-1$, we have

(*a*) $c_1=\text{Lt}\ (y-m_1x)=\text{Lt}\ (y+x)$, as x and $y\to\infty$.

Now $(y+x)(2y+x)=-(3x-2y+1)$ (*curve*)

i.e., $\quad y+x=-\dfrac{3x-2y+1}{2y+x}$

$$\therefore\ \text{Lt}\ (y+x)=-\text{Lt}\frac{3x-2y+1}{2y+x}=-\text{Lt}\ \frac{3-2\frac{y}{x}+\frac{1}{x}}{2.\ \frac{y}{x}+1}$$

$$\therefore\ \text{Lt}\ (y+x)=-\frac{3-2(-1)+0}{2(-1)+1}$$

since $\text{Lt}\ \dfrac{1}{x}=0$ and $\text{Lt}\ \dfrac{y}{x}=m=-1$

$\therefore\ c_1=5.$

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Substituting these values of m and c in $y=mx+c$, we get $y+x=5$ as one of the asymptotes.

(*b*) $c_2=\text{I.t}\ (y-m_2x)=\text{Lt}\left(y+\frac{x}{2}\right)=\frac{1}{2}\ \text{Lt}\ (2y+x)$

Now $2y+x=-\frac{3x-2y+1}{x+y}$

$\therefore\ \text{Lt}\ (2y+x)=-\text{Lt}\ \frac{3x-2y+1}{y+x}$

$$=-\text{Lt}\ \frac{3-2\frac{y}{x}+\frac{1}{x}}{\frac{y}{x}+1}=-\frac{3+2\frac{1}{2}+0}{-\frac{1}{2}+1}=-8$$

$\therefore\quad c_2=-4.$

Substituting for m_2 and c_2 in $y=mx+c$, we get

$2y+x+8=0$ as the second asymptote.

Remarks on the examples.

(1) Observe that when m has been found in the first step, $y-mx$ is a factor of the highest degree terms in the equation of the curve. Thus in Ex. 1, $y+x$ is a factor of the highest degree terms in the equation of the curve and in Ex. 2, both $y+x$ as well as $2y+x$ (corresponding to the two values of m) are factors of $x^2+3xy+2y^2$, the highest degree terms.

(2) From remark (1), we infer that in algebraic curves the asymptotes are parallel to the lines obtained by equating to zero the linear factors of the highest degree terms.

(3) In algebraic curves, the determination of the limit is facilitated by first dividing the equation by a power of x which is equal to the degree of the curve.

(4) It should be observed that the *process of solution can be considerably shortened* by finding the limit *on the right side only* and *omitting the word limit from the left side*, as this does not make any difference in the equation of the asymptote.

Thus in the case of Ex. 1, step (*iii*) could have been altogether omitted and step (*ii*) could have been written in the following manner :—

The asymptote is

$$y+x=\text{Lt}\ \frac{6x^2}{x^2-xy+y^2} \text{ when } x\to\infty.$$

i.e., $y+x=2$ is an asymptote.

(5) An asymptote is usually asymptotic to an *algebraic curve* both on the right as well as on the left of the y-axis, since it does not make any difference whether $x\to\infty$ or $x\to-\infty$. For *an exception to this*, see example 3 below.

In some cases, for $x\to+\infty$, there is no limit for $\frac{y}{x}$ but there is a limit when $x\to-\infty$, and the curve will be asymptotic to the asymptote on the left of y-axis.

This point is illustrated by the following example.

Ex. 3. Show that the negative direction of the x-axis is an asymptote to the curve $y=e^x$.

(*i*) We have $m=\text{Lt}\frac{y}{x}=\text{Lt}\ \frac{e^x}{x}=0$, when $x\to-\infty$;

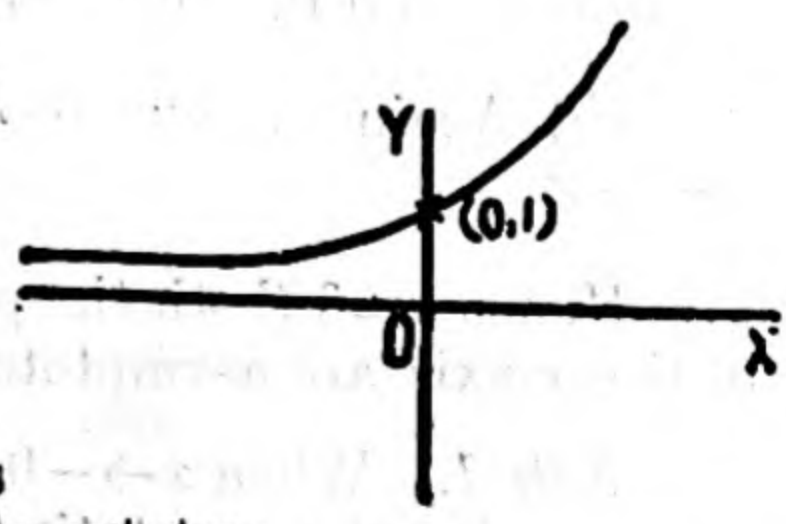
also $c=\text{Lt}\ (y-0.\ x)=\text{Lt}\ e^x=0$, when $x\to-\infty$.

$\therefore$ $y=0.x+0$ *i.e.*, $y=0$ or (x-axis) is an *asymptote.* Since these limits have been obtained for $x\to-\infty$, it is the *negative direction of x-axis which is* an asymptote.

(*ii*) Since $\text{Lt}\ \frac{y}{x}=\text{Lt}\ \frac{e^x}{x}$ for $x\to+\infty$ does not exist (this limit being infinity), the *positive direction of x-axis is no asymptote.*

The curve extends to infinity also in the positive direction of x but there is no asymptote on the right of y-axis.

Really there is an asymptote perpendicular to x-axis but at an infinite distance from the origin. But asymptotes at infinite distances from the origin will not be called asymptotes.



11. Asymptotes parallel to the axis of x and y.

The line $x=k$ is an asymptote to the curve $y=f(x)$, if $y\to\infty$ when x tends to k in a determinate sense.

For, if δ be the perpendicular from the point P(x, y) situated on the curve to the line $x=k$, $\delta=x-k$, (*ignoring the sign*), and this *perpendicular* tends to zero, (since x tends to k), as the point P moves to an infinite distance on the curve (since $y\to\infty$ under the condition that x tends to k). Hence we have the following **rule** for finding asymptotes to $f(x, y)=0$ which are parallel to the axis of y :—

Find the finite value or values of x which make y tend to ∞ ; if k_1, k_2 .....are such values, then the asymptotes parallel to y-axis are $x=k_1$, $x=k_2$.....

In a similar manner, we can find the **asymptotes parallel to the axis of x** *by the following* **rule** : *Find those values (finite) of y, which will make x tend to ∞ ; let these values be l_1, l_2, l_3 .. then the asymptotes parallel to the axis of x are $y=l_1$, $y=l_2$ .....*"

Ex. 1. Examine the curve $y+1=e^{\frac{1}{x}}$ for asymptotes parallel to co-ordinate axes.

We enquire if there is any number to which x tends, when $y\to\infty$.

We find that if $x\to+0$, $y=+\infty$.

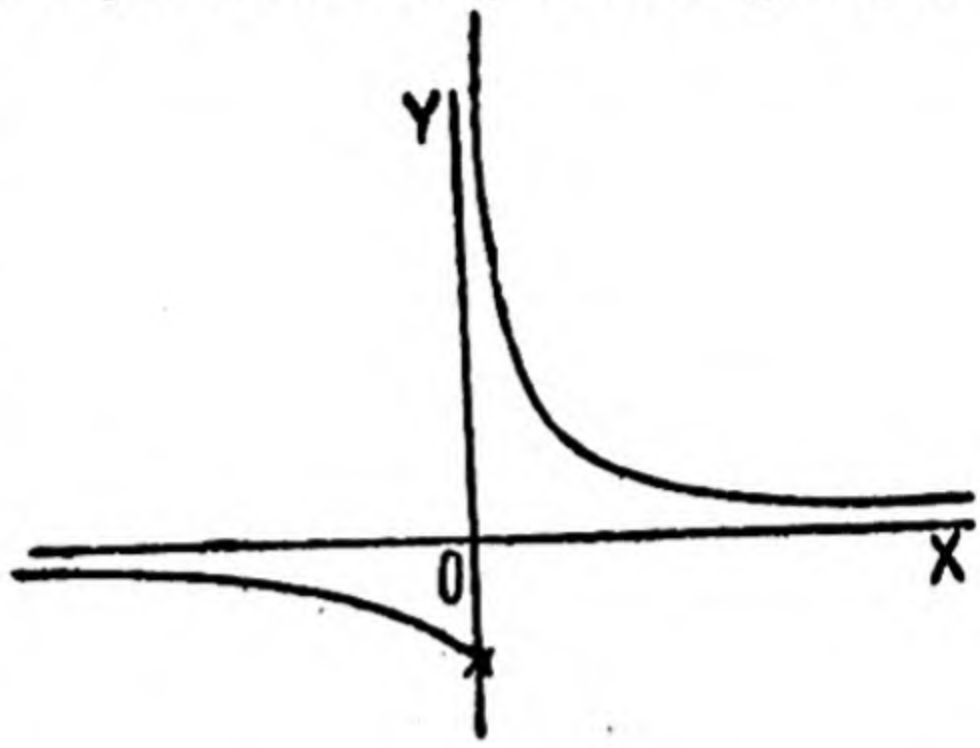
(*i*) $\therefore$ $x=0$ (the positive direction of the y-axis) is an asymptote and the curve is to the right of the asymptote.

(*ii*) Again when $y\to+0$, $x\to+\infty$, and when $y\to-0$, $x\to-\infty$.

Hence $y=0$ (both the positive and the negative directions of the x-axis are asymptotes) is an asymptote.

Note 1. When $x\to-0$, $y\to-1$, $\therefore$ the negative direction of x-axis is not asymptotic to the curve.

Note 2. In the case of an *algebraic curve* (as distinguished from *transcendental curves*), a rectilinear asymptote is usually asymptotic to the curve at both ends.



Ex. 2. Show that $x=2$, $x=4$ and $y=1$ are asymptotes to the curve

$$y=\frac{(x-1)(x-3)}{(x-2)(x-4)}.$$

12. Asymptotes—Algebraic Curves. We have already given practical methods of finding asymptotes of algebraic curves. We shall now indicate briefly that the same methods can be deduced from the general definition of an asymptote given in Art. 9.

13. Asymptotes parallel to the axis of y. Let the equation of the curve of degree n be $f(x, y)=0$. Rationalise and clear the equation of fractions; and then *arrange according to descending powers of y*. Thus, the equation $f(x, y)=0$ is expressed in the form

$$ay^n+(bx+c)y^{n-1}+(dx^2+ex+f)y^{n-2}+\ldots\ldots=0 \qquad (1)$$

the last term being a polynomial of the nth degree in x.

(*i*) Suppose the term y^n is present in equation (1). This means that the co-efficient "*a*" *is not zero*. Then when any finite value is given to x in (1), *no value of y is infinite*. There are, therefore, no infinite ordinates for finite values of the abscissa. Hence *in this case there are no asymptotes parallel to the axis of y*.

(*ii*) Next suppose that the term y^n is *missing from the equation* so that $a=0$. Suppose further that b *and* c *are not zero*.

(*a*) In this case one value of y is infinite for *every finite value of* x, giving a branch of the curve lying entirely at infinity. *We ignore this case, as it gives an asymptote situated at an infinite distance from the origin.*

(*b*) But in this case there is a particular value of x given by $bx+c=0$, *i. e.*, $x=-c/b$ which makes y infinity. Thus *the*

line $x=-c/b$ satisfies the condition for being an asymptote. Thus *the line* $bx+c=0$ *is an asymptote parallel to y-axis situated* at finite distance from the origin.

(*iii*) Suppose the terms involving y^n and y^{n-1} are both absent from (1) so that $a=b=c=0$.

In this case there are *two finite values of* x which make y infinite and these values satisfy the equation $dx^2+ex+f=0$. The *equation of the asymptotes parallel to y-axis are found by factorising* $dx^2+ex+f=0$, and so on generally.

14. Asymptotes parallel to the axis of x. Proceeding in a similar manner but arranging the equation of the curve according to descending powers of x in the form

$$a_1x^n+(b_1y+c_1)x^{n-1}+(d_1y^2+e_1y_1+f_1)x^{n-2}+\ldots\ldots$$

we can easily determine the asymptotes parallel to the x-axis by finding the values (finite) of y which give infinite values to x.

Thus we have proved the usual rules for finding asymptotes parallel to axes given in Art. 5.

15. Oblique Asymptotes. Let the equation of the curve of the nth degree be

$$x^n\phi_n\left(\frac{y}{x}\right)+x^{n-1}\phi_{n-1}\left(\frac{y}{x}\right)+x^{n-2}\phi_{n-2}\left(\frac{y}{x}\right)+\ldots$$

$$+x\phi_1\left(\frac{y}{x}\right)+\phi_0=0 \qquad \ldots\ldots(1)$$

(*i*) Now in order to find an asymptote, we must first find, from the equation of the curve, Lt y/x when x and y tend to ∞. We introduce a *new variable* $t=y/x$, so that equation (1) becomes

$$x^n\phi_n(t)+x^{n-1}\phi_{n-1}(t)+x^{n-2}\phi_{n-2}(t)\ldots\ldots+x\phi_1(t)+\phi_0=0$$

the left-hand side being a function of n and t only. Dividing, by x^n, we get

$$\phi_n(t)+\frac{1}{x}\phi_{n-1}(t)+\ldots\ldots+\frac{1}{x^{n-1}}\phi_1(t)+\frac{1}{x^n}\phi_0=0.$$

Supposing asymptotes to exist, t must approach a limit, say m when $x\to\infty$. Hence proceeding to the limit, when $x\to\infty$, all the terms, except the first on the left vanish and we get $\phi_n(m)=0$.

This equation gives the *directions of the asymptotes* and since $\phi_n(m)=0$ is, in general, an equation of the nth degree, there are generally n directions in which the curve can extend to infinity.

(*ii*) Let m be one of the roots of the equation (1).

In order to find the asymptote in this direction, we have next to find

Lt $(y-mx)$ when x and $y\to\infty$.

Introduce another new variable v and put

$$y-mx=v. \quad \text{Then } y/x=m+v/x.$$

Substituting for y/x in (1), we get

$$x^n\phi_n\left(m+\frac{v}{x}\right)+x^{n-1}\phi_{n-1}\left(m+\frac{v}{x}\right)+\ldots\ldots=0$$

Divide by x^n and expand by Taylor's Theorem ;

$$\phi_n(m)+\frac{1}{x}\left(v\phi'_n(m)+\phi_{n-1}(m)\right)+$$

$$\frac{1}{x^2}\left\{\frac{v^2}{|2}\phi''_n(m)+v\phi'_{n-1}(m)+\phi_{n-2}m)\right\}+\ldots\ldots=0.$$

Since $\phi_n(m)=0$ by (1), this becomes, after multiplying the terms by x.

$$\{v\phi'(m)+\phi_{n-1}(m)\}+$$

$$\frac{1}{x}\left\{\frac{v^2}{|2}\phi''_n(m)+v\phi'_{n-1}(m)+\phi_{n-1}(m)\right\}+\ldots.=0.$$

Suppose v, [*i. e.*, $(y-mx)$], approaches the limit c when $x\to\infty$, then

$$c\phi'_n(m)+\phi_{n-1}(m)=0 \qquad \ldots\ldots(II)$$

If $\phi_n'(m)\neq 0$, we have $c=-\dfrac{\phi_{n-1}(m)}{\phi'_n(m)}$ .....(III)

Thus, the equation of the asymptote corresponding to the direction m [m satisfying the equation $\phi_n(m)=0$] is

$$y=mx-\frac{\phi_{n-1}(m)}{\phi'_n(m)}.$$

16. Other definitions of rectilinear asymptotes.

It is usual to give the following *definitions* of asymptote *which are, however, inadequate.*

1. "An asymptote is defined as the limit of a tangent whose point of contact tends to infinity."

2. "An asymptote is the limiting position of a chord two of whose points of intersection with the curve together tend to infinity."

3. "An asymptote to an infinite branch of a curve is a straight line which intersects the given curve in two points at infinity but which is situated at a finite distance from the origin."

Asymptotes, however, may exist in the sense of the definition we have given when the tangent or chord has no limit ; the curve may not even have a tangent.

The precise relations between asymptotes and tangents are defined by the following theorems :

(α) "If a tangent to the curve $y=f(x)$, whose point of contact is P, has the limit $y=mx+c$ as $P\to\infty$, then $y=mx+c$ is an asymptote."

(β) "If a chord of the curve $y=f(x)$ through the points P and Q on the curve has a limit when $P\to\infty$, $Q\to\infty$ together in a direction in which the curve is asymptotic to its asymptote, then the limiting position of the chord is an asymptote."

The following illustrations would show that definitions 1, 2, 3 of this section are inadequate :—

Ex. 1. Consider the curve $y=Ax+B+\dfrac{a+\sin x}{x}$. (Todhunter)

To find an asymptote, we have to determine

I. (*i*) $\text{Lt}\ y/x=\text{Lt}\left(A+B/x+\dfrac{a+\sin x}{x^2}\right)=A$

and (*ii*) $\text{Lt}\ y-Ax=\text{Lt}\left(B+\dfrac{a+\sin x}{x}\right)=B,$

when x and y tend to ∞,
showing that $y=Ax+B$ *is the asymptote.*

Again, since the equation of the tangent at (x, y) is

$$Y=\frac{dy}{dx}X+\left(y-x\frac{dy}{dx}\right),$$

we have to determine, if we adopt definition 1 of this section for the definition of an asymptote.

II. (*iii*) Lt $dy/dx = \text{Lt}\left\{\text{A}+\dfrac{x(a+\cos x)-(a+\sin x)}{x^2}\right\}=\text{A}$

and (*iv*) Lt $(y-x.dy/dx)=\text{Lt}\left(\text{B}-\cos x+\dfrac{2(a+\sin x}{x}\right)$

when x and y tend to ∞.

But since cos x does not approach any limit when $x\to\infty$, (*iv*) has no limit. Therefore there is *no asymptote according to this definition.*

III. If we adopt definition 3, we have to substitute $mx+c$ for y ; this gives the identity $mx+c=\text{A}x+\text{B}+\dfrac{a+\sin x}{x}$ and we cannot get the value of m and c by equating to zero the coefficients of the highest powers of x. Thus, there is no asymptote according to definition 3 also.

Ex. 2. Consider the curve $y=cx+x^{-\alpha}\sin x^2$. (Young)

If $\alpha>0$, the curve has the asymptote $y=cx$, but $\dfrac{dy}{dx}$ has no limit as $x\to\infty$ unless $\alpha>1$.

If $\alpha>1$, $\dfrac{dy}{dx}-c=2x^{1-\alpha}\cos x^2-\alpha x^{-1-\alpha}\sin x^2\to 0$;

but $x\left(\dfrac{dy}{dx}-c\right)\to 0$ only if $\alpha>2$. In this case only ($\alpha>2$) *the tangent tends to the asymptote.*

It may be pointed out that these definitions give rectilinear asymptotes of algebraic curves, for the restrictions in (α) and (β) are satisfied.

Method of Expansion

17. We can assert that the straight line $y=mx+c$ is actually an asymptote of the curve $f(x, y)=0$, provided that we show that there exists a real branch of the curve of the form $y=mx+c+\phi(x)$ where $\phi(x)\to 0$ as $x\to\infty$ or $-\infty$.

Suppose it is possible to solve for y from the equation of given curve so as to express y in a series of descending powers of x, *beginning with the first power*, in the form

$$y=mx+c+\alpha/x+\beta/x^2+\gamma/x^3+\ldots\ldots \quad ..(1)$$

Then it is evident that $y=mx+c$ is an asymptote to the curve represented by (1). For

(i) $y/x=m+c/x+\alpha/x^2+\beta/x^3+\gamma/x^4+\ldots\ldots$

$\therefore$ Lt $y/x=m$, when x and y tend to infinity.

(ii) $y-mx=c+\alpha/x+\beta/x^2+\gamma/x^3+\ldots\ldots$

$\therefore$ Lt $(y-mx)=c$ when x and y tend to infinity.

Hence $y=mx+c$ *is an asymptote.*

Corollary. From the above we deduce the following :—

"A straight line will be an asymptote to a curve when the difference between the ordinates of the curve and of the line, corresponding to a common abscissa, approaches zero as a limit as the abscissa tends to infinity."

Note. This follows immediately from the definition of an asymptote, since the *distance* from any point of the curve on the asymptote *measured in any other direction* ($\therefore$ the distance measured parallel to y-axis) being proportional to the perpendicular distance (δ) on the asymptote must tend to zero, when the point goes to infinity.

18. The problem for finding an asymptote to a curve is thus reduced to the problem of expanding y from the equation of the curve in the form

(1) $y=mx+c+\alpha/x+\beta/x^2+\gamma/x^3+\ldots\ldots$

Two cases arise according as the equation of the curve is given as an **explicit** or **implicit** *function* of its co-ordinates.

Case I. When y is given *explicitly* in terms of x, the asymptotes are readily obtained by expanding y in descending powers of x by means of the Binomial Theorem. This method also applies when x is given explicitly in terms of y.

Case II. When the curve is given by the *implicit relation* $f(x, y)=0$ the expansion of y in decending powers of x (whenever such expansion is possible is readily effected by substituting for y the series on the right side of (1) in the equation $f(x, y)=0$, and *treating the resulting equation as an identity.* Equating then the co-efficients of different powers of x to zero, sufficient conditions are obtained for determining the constants $m, c, \alpha, \beta, \gamma\ldots\ldots$

19. *To find on which side of the asymptote the curve lies.*

When the equation of the curve has been thrown into the form

$$y=mx+c+\alpha/x+\beta/x^2+\gamma/x^3\ldots\ldots(curve)$$

$y=mx+c$ is an *asymptote.*

If the ordinates of the curve and of the asymptote for the same abscissa x be denoted by y_1 and y_2, then

$$y_1-y_2=\alpha/x+\beta/x^2+\gamma/x^3+\ldots\ldots$$

The sign of "y_1-y_2" will be determined by the first term α/x which governs the sign of the right hand side when x is sufficiently large. The *sign of* α (or in its absence the sign of β) *will decide whether the curve is above or below the asymptote in any particular quadrant through which the asymptote passes.*

Suppose the asymptote passes through the *first and the third quadrants. Suppose further that* α *is positive.* Then

(*i*) in the *first quadrant,*

$y_1-y_2=\alpha/x$ (positive), since x is $+$.

$\therefore$ *ordinate of curve* $>$ *ordinate of asymptote.*

$\therefore$ *the curve lies above the asymptote* ;

(*ii*) in the *third quadrant.*

$y_1-y_2=\alpha/x$ (negative) since x is $-$.

Also y_1 and y_2 are *both negative* in the third quadrant.

$\therefore$ y_2 *is numerically smaller* but *algebraically greater than* y_1.

Hence the *curve lies below the asymptote.*

Similar investigations will apply when α is negative and in all other cases.

Note. In actual practice, to throw the equation of the curve in the form (1), it is convenient

(*i*) *to first find an asymptote* $y=mx+c$ to $f(x, y)=0$ by one of the methods already explained, so that m *and* c *are now known quantities.*

(*ii*) The expression for y from (1) is then substituted in $f(x, y)=0$.

(*iii*) The co-efficients of the different powers of x in the equation resulting from this substitution are then put equal to zero to determine the *remaining constants* α, β, γ etc.

Ex. 1. Find the asymptotes of the hyperbola

$$x^2/a^2 - y^2/b^2 = 1$$

Solving for y, we get $y = \pm b/a.x(1 - a^2/x^2)^{\frac{1}{2}}$

$$= \pm b/a.x(1 - 1/2.a^2/x^2 - 1/8.a^4/x^4 \ldots\ldots)$$

$$\therefore \quad y = +\frac{b}{a}\, x - \frac{1}{2}\, b\, \frac{a}{x} - \frac{1}{8}\, b\, \frac{a^3}{x^3}$$

$$\text{or} \quad y = -\frac{b}{a}\, x + \frac{1}{2}\, b\, \frac{a}{x} + \frac{1}{8}\, b\, \frac{a^3}{x^3} \ldots\ldots$$

Here $c = 0$ and, therefore,

(*i*) $y = +b/a.x$ and (*ii*) $y = -b/a.x$ are the two asymptotes.

(A) Thus $y = +\frac{b}{a}\, x$ and $y = +\frac{b}{a}\, x - \frac{1}{2}\, \frac{ab}{x} \ldots\ldots$ are equations giving ordinates of asymptote (*i*) and of the corresponding branch of the curve.

(B) Similarly $y = -\frac{b}{a}\, x$ and $y = -\frac{b}{a}\, x + \frac{1}{2} \cdot \frac{ba}{x} + \ldots$ are equations giving ordinates of asymptote (*ii*) and of the corresponding branch of the curve.

Case (*i*) $x \to +\infty$; equations (A) show that ordinate of asymptote > ordinate of curve.

(*a*) Since $-ab/2\, x$ is negative, a and b being $+$,

$\therefore$ the curve lies below the 1*st asymptote*.

(*b*) Equations (B) show that *ordinate of curve minus ordinates of asymptote* $= +ab/2x + \ldots\ldots$

$\therefore$ the curve lies above the 2nd asymptote.

Case (*ii*). $x \to -\infty$ can be similarly treated.

Ex. 2. Find the oblique asymptotes of the curve

$$y^2 = x^2/(x-1).$$

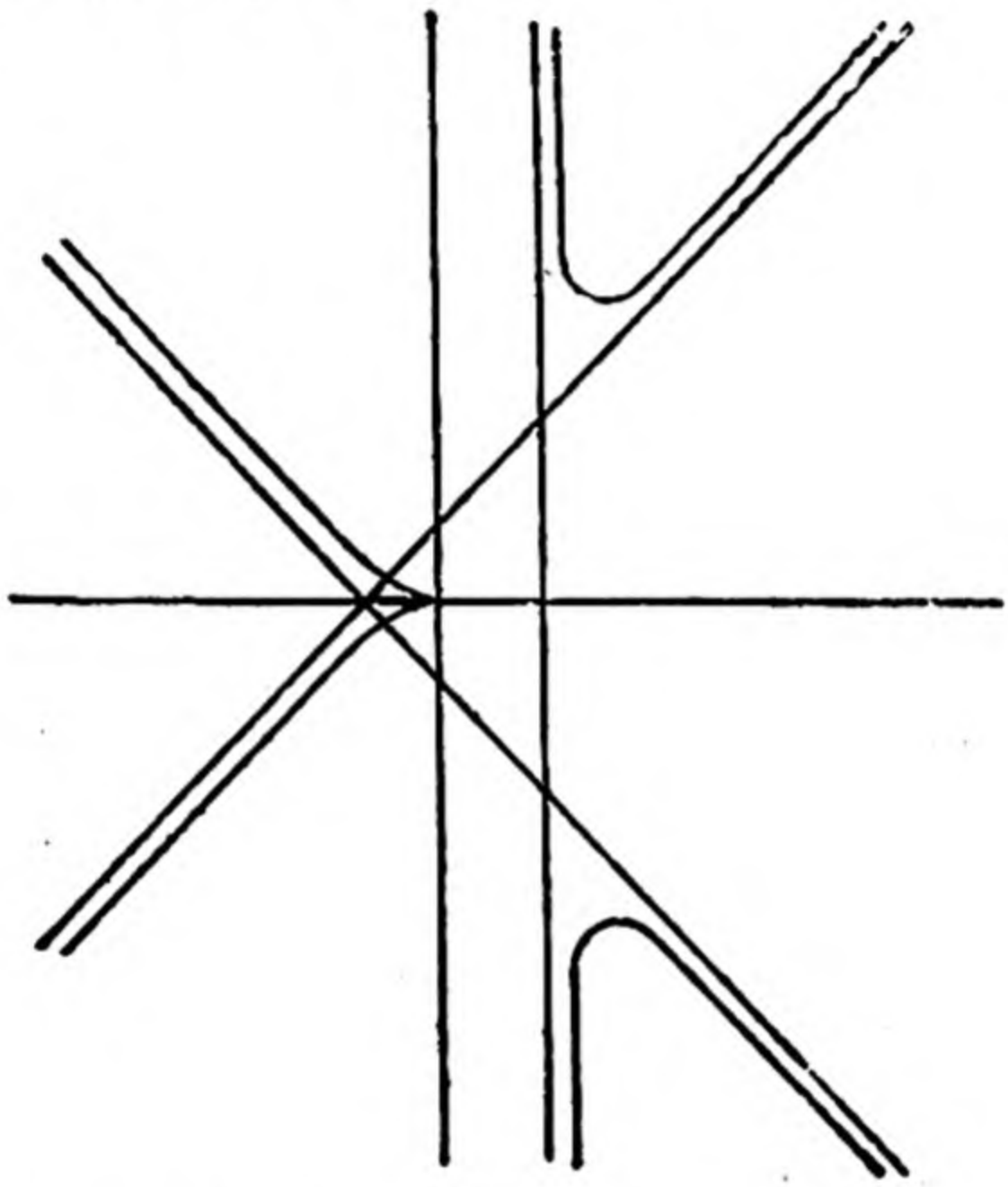
We have $y^2 = x^2(1 - 1/x)^{-1}$.

$\therefore \quad y=\pm x(1-1/x)^{-\frac{1}{2}}$

$=\pm x(1+1/2x+3/8x^2+5/16.1/x^3+\ldots\ldots)$

i. e., $y=+x(1+1/2x+3/8x^2+5/16x^3\ldots\ldots)$

or $y=-x(1+1/2x+3/8x^2+\ldots\ldots)$



$\therefore$ (*i*) $y=x+1/2$ and (*ii*) $y=-x-1/2$ are the two oblique asymptotes, the third asymptote being evidently $x-1=0$.

(A) Thus $y=x+1/2$ and $y=x+1/2+3/8x+5/16x^2+\ldots\ldots$ are the equations giving the ordinates of asymptote (*i*) and the corresponding branch.

(B) Similarly $y=-x-1/2$ and $y=-x-1/2-3/8x\ldots\ldots$ are the equations giving the ordinates of asymptote (*ii*) and the corresponding branch.

Case (*i*) For $x\to+\infty$:

(*a*) equations (A) show that the *ordinates of curve minus ordinate of asympotote* is +

$\therefore$ the curve is above the first asymptote.

(*b*) equations (B) show that *ordinate of asymptote minus ordinate of curve* $= +3/8x+5/16x^2\ldots\ldots$

$\therefore$ the curve *is below the asymptote.*

Case (*ii*) $x\to-\infty$; in this case the curve is below 1st asymptote but above the second.

Observe that the line $x=1$ is the third asymptote.

Ex. 3. Find the position with regard to the asymptote of the curve

$$x^3-y^3+a^2(x-2y)=0 \qquad \ldots\ldots(1)$$

$y-x=0$ is an asymptote.

$\therefore$ Putting $y=x+0+\alpha/x+\beta/x^2+\ldots\ldots$ (2)

in the equation of the curve, we get

$x^3-(x+\alpha/x+\beta/x^2+\ldots)^3+a^2\{x-2(x+\alpha/x+\beta/x^2+\ldots)\}=0.$

$-3x^2(\alpha/x+\beta/x^2)-3\alpha^2/x^2(x+\beta/x^2)-3\beta^2/x^4(x+\alpha/x)+\alpha^3/x^3$

$+\beta^3/x^6-6\alpha\beta/x^2+a^2x-2a^2x-3a^2\alpha/x-2a^2\beta/x^2=0$

i. e., $-x(3\alpha+a^2)-3\beta-3a^2/x-2a^2\alpha/x+\ldots=0$

Putting the co-efficients of several powers of $x=0$,

we get $\therefore\ 3\alpha+a^2=0 \qquad 3\beta=0$

$\ldots\ldots\ldots\ldots\ldots\ldots$

i. e., $\alpha=-a^2/3$ and $\beta=0$.

Hence substituting for α and β in (2), we have

$$y=x+0-a^2/3x+\ldots\ldots \qquad (curve)$$

also $y=x$ (*asymptote*)

ordinate of curve—ordinate of asymptote $=-a^2/3x$

(*i*) $\therefore$ when x is positive the curve lies *below the asymptote* $y-x=0$ in the *first quadrant.*

(*ii*) when x is negative.

ordinate of curve $>$ ordinate of asymptote.

Since both the ordinates are negative in the *3rd quadrant,* ordinate of curve $<$ ordinate of asymptote *numerically.*

$\therefore$ the curve lies *above the asymptote.*

Ex. 4. Find the position with regard to the asymptote of the curve $x^3+y^3=3axy$ ...(1)

$$(y+x)(y^2+x^2-xy)=3axy$$

$\therefore$ the asymptote is given by

$$y+x=\text{Lt}\ \frac{3axy}{y^2+x^2-xy}\ \Bigg|\ \begin{matrix}x,\ y\to\pm\infty\\ y\\ x\end{matrix}\ =-1$$

$$=-\text{Lt}\ \frac{3a.y/x}{y^2/x^2+1-y/x}=-a$$

$\therefore$ $y=-x-a$ is the only real asymptote.

Putting $y=-x-a+\alpha/x+\beta/x^2+\ldots\ldots$ ...(2)

in (1), we get

$$x^3+(-x-a+\alpha/x+\beta/x^2)^3=3ax.(-x-a+\alpha/x+\beta/x^2)$$

$$i.\ e.,\ x^3-x^3+(-a+\alpha/x+\beta/x^2)^3+3x^2(-a+\alpha/x+\beta/x^2)$$

$$-3x(-a+\alpha/x+\beta/x^2\ldots\ldots)^2=3ax(-x-a+\alpha/x+\beta(x^2+\ldots)$$

Equating the co-efficients of the several powers of x in this identity, we get

$$\left.\begin{matrix}3\alpha=0\\ 3\beta=a^3\end{matrix}\right]\quad\begin{matrix}\text{Equating co-efficient of } x.\\ \text{Equating the absolute terms.}\end{matrix}$$

Substituting for α, β in (2), we obtain

$y=-x-a+a^3/3.\ 1/x^2\ldots\ldots$ *(curve)*

$y=-x-a\ldots\ldots$ *(asymptote)*

Hence (*ordinate of curve*)—(*ordinate of asymptote*) $=\frac{a^3}{3}\cdot\frac{1}{x^2}$,

which is positive whether x be positive or negative.

$\therefore$ the curve lies *above the asymptote* both in the *second* as well as the *fourth quadrant* through which the asymptote passes.

20. Curvilinear asymptotes. In considering rectilinear asymptotes we have met with cases where the method of determining asymptotes in particular direction gives an infinite value to c; the asymptotes in this case are situated altogether at infinite and we say that there are no rectilinear asymptotes (as in the case of a parabola). In such cases we are sometimes able to find some other curve (or curves) which continually approaches the given curve as x, y tend to become infinite. Such asymptotes are called *curvilinear asymtpotes*.

Definition. If there be two curves which continually approach each other so that the difference of their ordinates corresponding to a common abscissa approaches zero as a limit when the common abscissa tends to become infinite, then each curve is said to be an *asymptote* to the other.

This definition will also apply if the words "ordinate" and " abscissa" be interchanged.

This definition is identical with that of asymptotic functions which we have already come across.

For example, the curves

$$y=ax^2+bx+c+d/x$$

and $$y=ax^2+bx+c$$

are *asymptotic* to each other, since the difference of their ordinates for any common abscissa x is d/x, which tends to zero as $x\to\infty$.

Since $y=ax^2+bx+c$ is a parabola, it is called a *parabolic asypmtote* to the curve $y=ax^2+bx+c+d/x$ (*3rd degree curve*).

[**Note**. If the equation to a curve can be put in the form

$$(A)\quad y=a_0x^n-a_1x^{n-1}+\ldots\ldots+a_{n-1}x+a_n + \frac{b_1}{x}+\frac{b_2}{x^2}+\frac{b_3}{x^3}+\ldots\ldots$$

Then it follows from the definition that the curves

(1) $y=a_0x^n+a_1x^{n-1}+\ldots\ldots+a_n$

(2) $y=a_0x^n+a_1x^{n-1}+\ldots\ldots+a_n+b_1/x$

(3) $y=a_0x^n+a_1x^{n-1}+\ldots\ldots+a_n+b_1/x+b_2x^2$

$\ldots\ldots\ldots\ldots\ldots\ldots$

are all asymptotes to the curve (A).

(They may be called asymptotes of different degrees.)

This shows that if a given curve has *curvilinear asymptotes* the number of such asymptotes is generally infinite.

Ex. Show that the curve $ay(a-x)=x^3$ has a parabolic asymptote.

We have $ay=x^3/(a-x)=-x^2-ax-a^2+a^3/(a-x)$

$$=-x^2-ax-a^2-\frac{a^3}{x}\left(1-\frac{a}{x}\right)^{-1}$$

$$=-x^2-ax-a^2-a^3/x(1+a/x+a^2/x^2+a^3/x^3+\ldots)$$

Hence the equation of the curve is

$$ay=-x^2-ax-a^2-a^3/x-a^4/x^2-a^5/x^3-a^6/x^4\ldots$$

$\therefore$ $ay=-x^2-ax-a^2$ is the *curvilinear asymptote* which is a parabola,

i. e., $x^2+a(y+x)+a^2=0$ is the parabolic asymptote.

EXAMPLES XXXIV

1. Show that the curve $axy=x^3-a^3$ has a parabolic asymptote $ay=x^2$.

2. Show that $ay^2=x(x-a)(x-2a)$ has a curvilinear asympote $ay^2=(x-a)^2$.

3. Show that the curve $y^2(2y+3x)=4y^2+9x^2$ has a parabolic asymptote $y^2=3x-2y+8$.

[*Hint.*—Observe that one form of the curve at infinity is

$$3xy^2=9x^2,$$

i. e., $y^2=3x$ (Newton's diagram of powers) so that y^2 is proportional to x. Therefore, arranging the terms in order of importance, we have

$$y^2(3x+2y)=9x^2+4y^2$$

$$i.\ e.,\qquad y^2=\frac{9x^2+4y^2}{3x+2y}=3x-2y+8$$

the last remainder $8y^2$ being put equal to $24x$.]

Note. Observe that in the case of *curvilinear asymptotes* y and x are *of different orders of infinity* whereas in the case of *rectilinear asymptotes* (except in the case of asymptotes parallel to co-ordinate axes where one of the co-ordinates is finite and the other infinite) y and x are of the *same order of infinity* so that the ratio y/x is finite.

4. Show that the curve $x^3+aby=axy$ has a parabolic asymptote $x^2+bx+b^2=ay$.

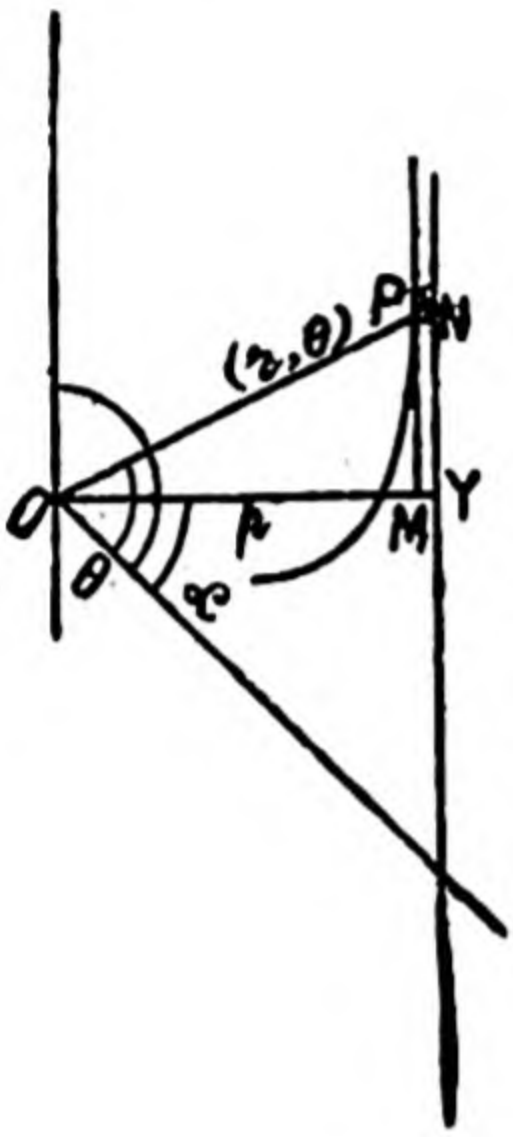
21. Asymptotes Polar Co-ordinates.

Let $P(r, \theta)$ be a point on the curve $r=f(\theta)$ and δ, the perpendicular from the point P on the line whose polar equation is $r\cos(\theta-\alpha)=p$. Then if $\delta\to 0$ as $r\to\infty$, the line $r\cos(\theta-\alpha)=p$ is by definition, an asymptote to the curve $r=f(\theta)$.

We proceed to determine the values of α and p in order that $r\cos(\theta-\alpha)=p$ may satisfy the conditions of being an asymptote to the curve $r=f(\theta)$.

The length of the perpendicular from P to the line is

easily seen from the figure to be $=OY-OM=p-r\cos(\theta-\alpha)$ so that



$\delta \equiv p-r\cos(\theta-\alpha)$

$\therefore\ \delta/r=p/r-\cos(\theta-\alpha)$.....(1)

When $r\to\infty$, $\delta\to 0$, (*definition*)

$\therefore\ \delta/r\to 0$ as well as $p/r\to 0$,

Hence from (1) Lt $\cos(\theta-\alpha)=0$ when $r\to\infty$.

i. e., Lt $(\theta-\alpha)=\pi/2$ when $r=\infty$.

(*i*) $\alpha=\theta_1-\pi/2$ where θ_1 corresponds to $r\to\infty$ on the curve. We have to let $r\to\infty$ in the equation to the curve and find the direction θ_1. Thus θ_1 gives the direction of the asymptote; also α is 90° less than this angle as is evident from the figure.

(*ii*) Again $p=OY=$polar subtangent in the direction $\theta=\theta_1$

$$=-\left(\frac{d\theta}{du}\right)_{\theta\to\theta_1}\left[=\left(r^2\frac{d\theta}{dr}\right)_{\substack{\theta\to\theta_1\\ r\to\infty}}\right],$$

where $\dfrac{d\theta}{du}$ is to be obtained from the equation to the curve by differentiation.

(*iii*) Substituting for α and p from (*i*) and (*ii*) in the equation $r\cos(\theta-\alpha)=p$, we get

$$r\cos\left[\ \theta-\left(\theta_1-\frac{\pi}{2}\right)\right]=\left(\frac{d\theta}{du}\right)_{\theta\to\theta_1},$$

i. e., $r\sin(\theta_1-\theta)=-\left(\dfrac{d\theta}{du}\right)_{\theta\to\theta_1}$ as the equation of the asymptote in the direction $\theta=\theta_1$.

If θ_2, θ_3, θ_4.....are the other values of θ obtained by making the radius vector infinite in the equation $r=f(\theta)$

of the curve, then the equations of the other asymptotes are

$$r \sin(\theta_2-\theta)=\left(-\frac{d\theta}{du}\right)_{\theta\to\theta_2}$$

$$r \sin(\theta_3-\theta)=\left(-\frac{d\theta}{du}\right)_{\theta\to\theta_3} \text{ and so on.}$$

Ex. 1. Find the asymptotes of the curve

$$r=a \sec\theta+b \tan\theta.$$

When $r=\infty$, $\cos\theta=\theta$ $\therefore$ $\theta=2k\pi\pm\pi/2$

(*i*) *i. e.*, $\theta=\pi/2, 3\pi/2, 5\pi/2$.....(for $k=0, 1, 2, \ldots$ .....) showing that the asymptotes are perpendicular to the initial line.

Again $u=\dfrac{\cos\theta}{a+b\sin\theta}$ $(u=1/r)$

$\therefore$ $\dfrac{du}{d\theta}=-\dfrac{(a\sin\theta+b)}{(a+b\sin\theta)^2}$

(*ii*) *i. e.*, $\left(-\dfrac{d\theta}{du}\right)=\dfrac{(a+b\sin\theta)^2}{a\sin\theta+b}$

(A) Now consider the direction $\theta=\pi/2$ from (*i*)

$$\left(-\frac{d\theta}{du}\right)_{\theta=\pi/2}=a+b \text{ (\textit{positive}) from (\textit{ii}).}$$

The asymptote is, therefore, perpendicular to the initial line at a distance $a+b$ to the *right of the pole*. Its *equation* is $r\sin(\pi/2-\theta)=a+b$.

(B) Consider the direction $\theta=3\pi/2$.

Corresponding polar subtangent $\left(-\dfrac{d\theta}{du}\right)_{\theta=3\pi/2}$

$a-b$ from (*ii*)

$\therefore$ its *equation* is $r\sin(3\pi/2-\theta)=a-b$.

These are the only *two different* asymptotes; asymptotes in the other directions coinciding with the one or the other.

Ex. 2. Show that the curve $r=a/(1-\cos\theta)$ has no asymptote.

When $r\to\infty$, $\cos\theta=1$ and $\theta=2k\pi$.

Again $u=(1-\cos\theta)/a$.

$\therefore\ du/d\theta=\sin\ \theta/a$

$$\therefore \left(-\frac{d\theta}{du}\right)_{\theta=2k\pi}=\left|-\frac{d}{\sin\theta}\right|_{\theta=2k\pi},$$

showing that $(-d\theta/du)\theta\to 2k\pi$ does not tend to finite limit *i.e.*, the polar subtangent is infinity and therefore the curve has no asymptote.

Ex. 3. Find the asymptotes of the curve

$$r\sin 4\theta=a\sin 3\theta$$

i.e., $$u=\frac{\sin 4\theta}{a\sin 3\theta}\left(u=\frac{1}{r}\right).$$

Putting $u=0$, we get $\sin 4\theta=0$

$\therefore\ 4\theta=k\pi+(-1)^k\times 0\ ;\ \theta=k.\ \pi/4$

(*i*) *i e.*, $$\theta=\frac{\pi}{4},\ \frac{\pi}{2},\ \frac{3\pi}{4},\ \pi,\ \frac{5\pi}{4}\ldots k=1,\ 2\ldots$$

Again $$\frac{du}{d\theta}=\frac{4a\sin 3\theta\cos 4\theta-3a\sin 4\theta\cos 3\theta}{(a\sin 3\theta)^2}$$

(*ii*) $\therefore\ \dfrac{d\theta}{du}=-\dfrac{(a\sin 3\theta)^2}{4a\sin 3\theta\cos 4\theta-3a\sin 4\theta\cos 3\theta}$

(A) Consider the direction $\theta=\pi/4$.

$\left(-\dfrac{d\theta}{du}\right)$ for $\theta=\dfrac{\pi}{4}=\dfrac{a}{4\sqrt{2}}$ (positive if a is positive).

(Hence the asymptote is parallel to a line drawn through the pole making an angle $\pi/4$ with the initial line and at a distance $\dfrac{a}{4\sqrt{2}}$ to the right of the pole perpendicular to the radius vector.)

Its *equation* is $r\sin\left(\dfrac{\pi}{4}-\theta\right)=\dfrac{a}{4\sqrt{2}}$.

The determination of the other asymptotes is left as an exercise to the student.

22. Circular asymptotes. In many polar curves when θ is made larger and larger without limit, the radius vector r approaches a definite limit "a" and the equation ultimately takes the form of a circle whose radius is "a". This circle is called an *asymptotic circle* (curvilinear asymptotes).

Ex. Find the asymptotic circle of the curve

$$r=a(\theta-\alpha)/(\theta+\alpha).$$

The curve may be written as $r=a\frac{1-a/\theta}{1+a/\theta}$.

As θ increases and tends to become infinite, r continually tends to the limit a, showing that $r=a$ is the asymptotic circle r is less than a when θ is positive $\therefore$ curve lies inside the asymptotic circle for positive values of θ. When θ is negative $r>a$ $\therefore$ the asymptotic circle lies within the curve.

EXAMPLES XXXV

Find the asymptotes of :

1. (*i*) $r^2\theta=a^2$. (*ii*) $r\sin 2\theta=a$.

2. $r\cos\theta=a\sin^2\theta$. **3.** $r\sin n\theta=a$.

4. Prove that $r\sin\theta=a$ is the only asymptote of the curve $r=a\operatorname{cosec}\theta+b$.

5. $r\sin 2\theta=a\cos 3\theta$. **6.** $r=a(\sec\theta+\cos\theta)$.

7. $r^n\sin n\theta=a^n$. **8.** $r\theta=a$. **9.** $r=a+b\cot n\theta$.

10. $r\cos 2\theta=a\sin 3\theta$.

11. Show that in the curve $r\theta=a$ the perpendicular from the point (r, θ) on the initial line OX is equal to $a\sin\theta/\theta$ and then show that the curve has an asymptote parallel to OX and at a distance a from OX.

12. Find the circular asymptote to the curve

$$r=(a\theta+b)/(\theta+d).$$

13. Find the equation to the pair of tangents from the point (p, α) to the hyperbola $l/r=1+e\cos\theta$.

Deduce the equation to its asymptotes.

14. Find the *circular asymptotes* of :

(*i*) $r=c\theta/(\theta+b)$ (*ii*) $r=a\theta^2/(\theta^2\pm b^2)$

(*iii*) $r=a\theta/(\theta-1)$ (*iv*) $r=(\theta+\cos\theta)/(\theta+\sin\theta)$.

15. If $r\to\infty$ as $\theta\to\alpha$ and if $f(\theta)(\alpha-\theta)\to a$ show that the straight line $r\sin(\alpha-\theta)=a$ is an asymptote to the curve $r=f(\theta)$.

Prove its converse also.

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CHAPTER X

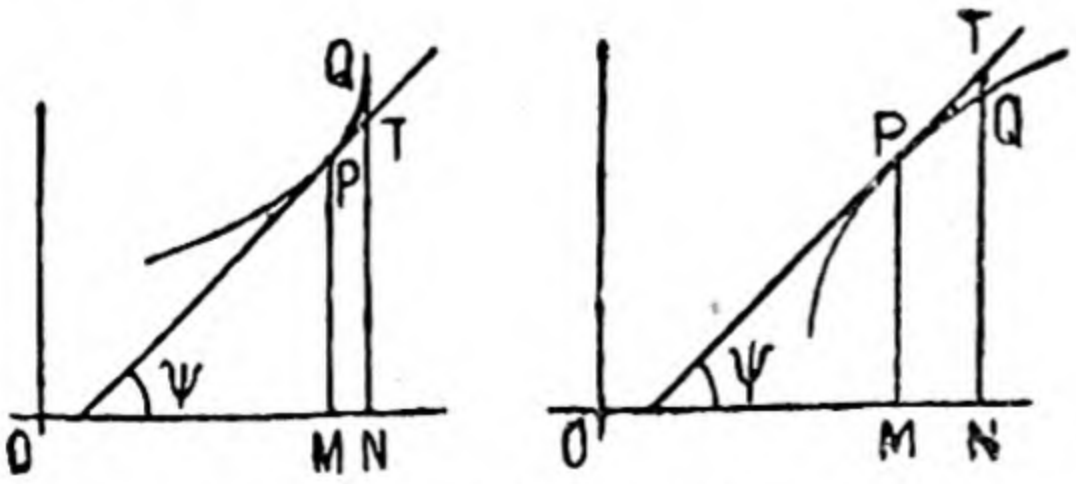
POINTS OF INFLEXION AND SINGULAR POINTS OF PLANE CURVES

1. Concavity. Let $y=f(x)$ be the equation of a curve and let P be a point on this curve the tangent at which is not parallel to the axis of y. If the curve does not cross its tangent at the point P (and this is usually the case) then it is situated only on one side of the tangent PT. Then we say that the curve is *concave* towards the side of positive y or the side of negative y according as the curve is situated with regard to the tangent on the side of positive y or negative y.

It may be remarked that there are two sides to a tangent which is not parallel to the y-axis. On one side y tends to $+\infty$ along any line parallel to y-axis while on the other y tends to $-\infty$; we have called the former the side of positive y and the latter the side of negative y.

2. Test of Concavity. We shall now show that at any point P the curve $y=f(x)$ is concave towards the side of positive y or negative y according as $f''(x)$ is positive or negative at that point.

In fact one or the other of those two cases happens according as the ordinate of the curve is greater than the ordinate of the tangent or less than that ordinate in the vicinity of that point.



Let P and Q be two neighbouring points on the curve $y=f(x)$ and the coordinates of P and Q be $[x, f(x)]$, $[(x+h), f(x+h)]$; the equation of the tangent at P is $Y=f(x)+(X-x)\,f'(x)$.

If y_2 be the ordinate of the tangent corresponding to the abscissa $x+h$, then

$$y_2=f(x)+hf'(x) \text{ (where } y_2=\text{NT}). \tag{1}$$

If y_1 (=NQ) be the ordinate of the curve corresponding to the same abscissa $x+h$, then

$$y_1=f(x+h)=f(x)+hf'(x)+\frac{h^2}{\underline{2}}f''(x+\theta h),\ 0<\theta<1 \tag{2}$$

by Taylor's Theorem, (3)

$$\therefore \quad y_1-y_2=\text{QT}=\frac{h^2}{\underline{2}}f''(x+\theta h). \tag{4}$$

Now, whether h is positive or negative, $\frac{h^2}{\underline{2}}$ is always positive and $f''(x+\theta h)$ keeps the same sign as that of $f''(x)$ for sufficiently small values of h provided that $f''(x)$ is continuous and not zero.

Therefore, the difference y_1-y_2 will retain its sign unchanged in the neighbourhood of P whether h is positive or negative (*i.e.*, whether Q is on the right or left of P).

Hence, throughout the neighbourhood of P the curve will lie wholly on one side of the tangent.

The curve will be below the tangent when $f''(x)$ is negative and above it when $f''(x)$ is positive.

Hence the result.

3. Points of inflexion. A point of inflexion is a point at which the curve crosses its tangent.

From equation (3) Art. 2 for the curve $y=f(x)$ where $f'(x)$ is continuous, we have

$$y_1-y_2=\frac{h^2}{\underline{2}}f''(x+\theta h),\ 0<\theta<1.$$

The necessary and sufficient condition that x may be a point of inflexion is that y_1-y_2, *i.e.*, $f''(x+\theta h)$ should change sign with h.

Hence at a point of inflexion $f''(x)$ vanishes and changes sign and conversely.

A sufficient condition that $f''(x)$ should change sign is $f'''(x)$ should not vanish with $f''(x)$. For when $f''(x)$ is zero but $f'''(x)$ is not zero, then

$$y_1-y_2=\frac{h^3}{\underline{3}}f'''(x+\theta_1 h),\ 0<\theta_1<1.$$

Since $f'''(x)$ is not zero, $f'''(x+\theta h)$ has the same sign as $f'''(x)$; but h^3 changes sign from $-$ to $+$, as h changes from $-$ to $+$. It follows therefore, that y_1-y_2 changes sign as we go from left to right through P. **There is thus a point of inflexion at P if $f''(x)=0$ and $f'''(x)$ be finite and not zero.**

But if $f'''(x)=0$, then it requires a further investigation. Suppose that the first derivative which is not zero is of order n. Then by Taylor's Theorem

$$y_1-y_2=\frac{h^n}{\lfloor n}f^n(x+\theta h).$$

The nth derivative, supposed continuous, retains the same sign for sufficiently small values of h. If n is odd then y_1-y_2 changes sign with h. Hence we get the following rule : In order that a root of $f''(x)=0$ may give a point of inflection, it is necessary that the first of those derivatives of order greater than second which is not zero should be of odd order.

However when such a derivative is of even order the point P is called a *point of undulation.*

Ex. 1. Examine for points of inflexion the curve $y=x^3$.

Here $\frac{dy}{dx}=3x^2$. $\frac{d^2y}{dx^2}=6x$; $\therefore$ $\frac{d^2y}{dx^2}=0$ when $x=0$

$\frac{d^2y}{dx^2}$ is negative when $x<0$ and positive when $x>0$.

Hence the origin is *a point of inflexion* since $\frac{d^2y}{dx}$ changes sign as x passes through zero. The curve is *concave on the left* and *convex on the right* of y-axis.

Ex. 1. Find the points of inflexion, in any, on the curve $y=x^4$.

(*i*) Here $y=x^4$; $\therefore$ $\frac{d^2x}{dy^2}=12x^2$.

Putting $\frac{d^2y}{dx^2}=0$, we find $x=0$;

$\therefore$ there is a possibility of a point of inflexion at the origin.

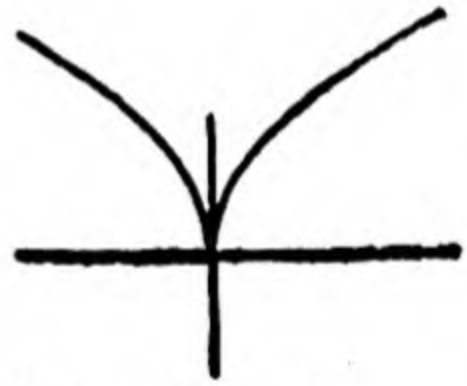
(*ii*) origin will be a point of inflexion if $\frac{d^2y}{dx^2}$ changes sign as x *passes through zero.* Giving to x a value slightly less than zero, we find $\frac{d^2y}{dx^2}$ is $+ve$; again giving to x a value slightly greater than zero, we find that $\frac{d^2y}{dx^2}$ is $+ve$. Since $f''(x)$ does not change sign as x passes through zero, therefore *origin is not a point of inflexion.*

This shows that the condition $\frac{d^2y}{dx^2}=0$ $\left(\text{or } \frac{d^2y}{dx^2}=\infty\right)$ is not sufficient for the existence of a point of inflexion.

Ex. 3. Examine for inflexion, the curve $y=x^{\frac{2}{3}}$.

Here $y=x^{\frac{2}{3}}$; $\frac{dy}{dx}=\frac{2}{3}x^{-\frac{1}{3}}$

and $\frac{d^2y}{dx^2}=-\frac{2}{9}\cdot\frac{1}{x^{\frac{4}{3}}}$



(*i*) Putting $\frac{d^2y}{dx^2}=\infty$, we get $x=0$.

$\therefore$ there is a possibility of a point of inflexion at $x=0$.

(*ii*) when x is slightly less than zero, $\frac{dy^2}{dx^2}$ *is negative* ; when x is slightly greater than zero, $\frac{d^2y}{dx^2}$ *is again negative.*

Hence *origin is not a point of inflexion since* $\frac{d^2y}{dx^2}$ *does not change sign at* $x=0$.

Ex. 4. Examine the curve $y=\frac{x^3}{a^2+x^2}$ for points of inflexion.

(*i*) Here, $f(x)=\frac{x^3}{a^2+x^2}$; $f'(x)=\frac{x^2(3a^2+x^2)}{(a^2+x^2)^2}$

$$f''(x)=\frac{2a^2x(3a^2-x^2)}{(a^2+x^2)^3}$$

(*ii*) $\therefore$ Putting $f''(x)=0$, we get

$x=0$; $x=+a\sqrt{3}$; $x=-a\sqrt{3}$.

There is possibility of points of inflexion where $x=0$, $x=\pm a\sqrt{3}$.

(*iii*) When x is slightly less than 0, $f''(x)$ is $-$; when x is slightly greater than 0, $f''(x)$ is $+$. $\therefore$ origin is a point of inflexion since $f''(x)$ changes sign.

Again in the neighbourhood of $x=+a\sqrt{3}$, $f''(x)$ is $+$ when $x<a\sqrt{3}$ and it is $-$ when $x>a\sqrt{3}$, $\therefore$ for $x=a\sqrt{3}$, there is a point of inflexion.

Similarly for $x=-a\sqrt{3}$, there is a point of inflexion.

4. If a curve has a point of inflexion at a point at which the tangent is parallel to the y-axis $\frac{d^2y}{dx^2}$ does not exist at such a point. To include this case, we say that for a curve for which one at least of $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{d^2y}$ exists and is continuous, the points of inflexion are the points at which one at least of $\frac{d^2y}{dx^2}$ and $\frac{d^2x}{dy^2}$ vanishes and changes sign.

5. Concavity and Convexity : Polar-co-ordinates.

A curve referred to polar co-ordinates is said to be *concave or convex to the pole at any point*, according as the curve in the neighbourhood of that point does or does not lie on the same side of the tangent as the pole, the pole being assumed not lying on the tangent.

(*i*) Let p be the perpendicular from the pole on the tangent at a point (r, θ), then if the curve is *concave to the pole*, p increases as r increases and p decreases as r decreases. (Fig. 1).

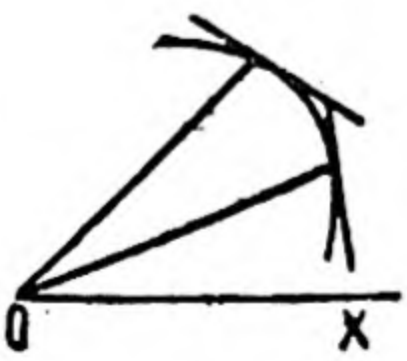


Fig. 1.

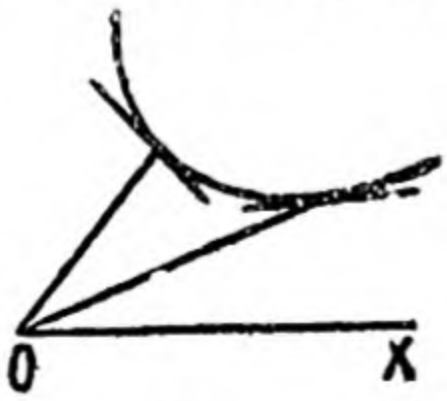


Fig. 2.

Thus $\frac{dp}{dr}$ is positive.

Similarly, if the curve be *convex to the pole*, $\frac{dp}{dx}$ *must be negative* (*Fig. 2*).

Hence *at a point of inflexion,* $\frac{dp}{dr}$ *must be zero and must also change sign.*

But $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$

Differentiating with respect to θ,

$$-\frac{2}{p^3}\frac{dp}{d\theta} = \frac{d}{d\theta}\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\} = \frac{d}{d\theta}(u)^2 + \frac{d}{d\theta}\left(\frac{du}{d\theta}\right)^2$$

$$= \frac{d}{du}(u^2).\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}$$

$$\therefore \quad -\frac{1}{p^3}\frac{dp}{\theta d} = \left(u+\frac{d^2u}{d\theta^2}\right)\frac{du}{d\theta}$$

$\therefore$ multiplying by p^3 and dividing by $\frac{du}{d\theta}$,

$$\frac{dp}{du} = -p^3\left(u+\frac{d^2u}{d\theta^2}\right) \qquad \ldots\ldots(1)$$

But $\frac{dp}{dr} = \frac{dp}{du}\cdot\frac{du}{dr} = -\frac{1}{r^2}\frac{dp}{du} = \frac{p^3}{r^2}\left(u+\frac{d^2u}{d\theta^2}\right).$

Since p is always taken positively, the curve is, in general, concave or convex to the pole according as $u+\frac{d^2u}{d\theta^2}$ is positive or negative; and at *a point of inflexion, the expression* $u+\frac{d^2u}{d\theta^2}$ *must be zero and must also change sign in passing through the point*

Ex. Find the point of inflexion in the curve

$$r = a\theta^2/(\theta^2-1).$$

Here $au = 1-\frac{1}{\theta^2} \quad \therefore \quad a\frac{du}{d\theta} = \frac{2}{\theta^3}$ and $a\frac{d^2u}{d\theta^2} = -\frac{2.3}{\theta^4}$

$$\therefore \quad a\left(u+\frac{d^2u}{d\theta^2}\right) = 1-\frac{1}{\theta^2}-\frac{2\cdot3}{\theta^2}$$

Setting this equal to zero, we get $\theta^4-\theta^2-6=0$, or $(\theta^2-3)(\theta^2+2)=0$ $\therefore$ at $\theta^2=3$ (*i. e.*, at $r=3a/2$), $\theta=\pm\sqrt{3}$, there will be a point of inflection if $u+\frac{d^2u}{d\theta^2}$ also changes sign. Now for θ^2 slightly <3, $u+\frac{d^2u}{d\theta^2}$ is $+$ and for θ^2 slightly >3,

$u+\dfrac{d^2u}{d\theta^2}$ is negative ; hence the curve has two points of inflexion at $r(=3a/2, \theta=\pm\sqrt{3})$.

Singular Points

6. A rigorous study of singular points is very difficult and beyond the scope of this book. We shall however let the reader have some working knowledge about these points.

The points at which the curve $f(x, y)=0$ possesses some peculiar property or properties such as stationary tangent, a point of inflexion, may be regarded to be singular points on the curve, but this name is usually reserved for a particular class of points at which $\dfrac{\partial f}{\partial x}=\dfrac{\partial f}{\partial y}=0$.

We will assume the existence and continuity of all the partial derivatives of $f(x, y)$ in the neighbourhood of the singular point under discussion.

7. We know that the direction of the tangent at an ordinary point (x, y) of the curve $f(x, y)=0$ is given by

$$\frac{dy}{dx}=-\frac{\partial f}{dx}\Big/\frac{\partial f}{\partial y}.$$

But at a singular point $\dfrac{\partial f}{dx}\Big/\dfrac{\partial f}{\partial y}$ is indeterminate.

In order to find the appearance of a curve in the neighbourhood of a singular point (x, y), we determine $\dfrac{dy}{dx}$ by *evaluating the indeterminate form*

$$\frac{dy}{dx}=-\frac{\dfrac{\partial f}{\partial x}}{\dfrac{\partial f}{\partial y}}\left(\frac{0}{0}\text{ form.}\right)$$

$$\text{Thus } \frac{dy}{dx}=-\frac{\dfrac{d}{dx}\dfrac{\partial f}{dx}}{\dfrac{d}{dx}\dfrac{\partial f}{\partial y}}=-\frac{\partial^2 f/\partial x^2+\partial^2 f/\partial x\partial y.dy/dx}{\partial^2 f/\partial x\partial y+\partial^2 f/\partial y^2.dy/dx}. \qquad (\Lambda)$$

Simplifying, we get the *quadratic equation*

$$\frac{\partial^2 f}{dy^2}\cdot\left(\frac{dy}{dx}\right)^2+\frac{2\partial^2 f}{\partial x dy}\cdot\left(\frac{dy}{dx}\right)+\frac{\partial^2 f}{\partial x^2}=0 \qquad (1)$$

to determine the slopes of the tangents at the singular point (x, y).

If $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial x \partial y}$, $\frac{\partial^2 f}{\partial y^2}$ be not simultaneously zero, the quadratic equation (1) will give *two values of* $\frac{dy}{dx}$ consequently there will be *two tangents* at (x, y) corresponding to the *two branches of the curve* passing through such a point which is called a *double point*.

Note. If $\frac{\partial^2 f}{\partial x^2}=0$, $\frac{\partial^2 f}{\partial x \partial y}=0$, $\frac{\partial^2 f}{\partial x^2}=0$ $\frac{dy}{dx}$ is still indeterminate in form and its value is to be determined by *differentiating separately* the numerator and denominator of (A). This will give *cubic equation in* $\frac{dy}{dx}$ giving three *directions* for the tangents ; thus *three branches of the curve pass through the point* and the point is called a *triple point* unless all the third partial derivatives in addition vanish simultaneously.

Double and triple points are examples of *multiple points* which are singular points on a curve through which more branches than one of the curve pass.

8. Double point. A singular point on a curve through which two real or imaginary branches of the curve pass is called a *double point.* The *curve, therefore, must have two tangents real or imaginary at a double point*, one belonging to each branch of the curve passing through it. Double points are classified into *nodes, cusps and conjugate points.*

If the tangents are real and different, the double point is called a **node** ; if the tangents are real and coincident, it is called a **cusp** ; if the tangents are imaginary, it is called a **conjugate point**. The general shape of the curve at a node and at a cusp is exhibited below.

Node | Cusp of the *first species* | Cusp of the *second species*

A *conjugate point* on a curve is an *isolated point* whose co-ordinates satisfy the equation of the curve but in its neighbourhood there is no other point whose co-ordinates satisfy the equation. This explains the absence of a real tangent at such a point.

Now, equation (1) gives the slopes of the two tangents at a double point and these tangents are *real an l different real and coincident*, or *imaginary*, according as

$$H \equiv \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} > = \text{or} < 0.$$

Accordingly, we deduce that *a double point is a node or a cusp in general or a conjugate point according as H is a positive, zero or negative.*

The above facts can be summarised into the following **rule** *for finding double points on the curve* $f(x, y)=0$ *and for determining their nature* :—

(*i*) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

(*ii*) Set $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$. Solve for x and y and test whether these values also satisfy $f(x, y)=0$. If real values of x, y can be found *which simultaneously satisfy* the *three equations* $f(x, y)=0$, $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$, then the points so determined are double points on the curve.

(*iii*) To determine the nature of a double point, calculate the value of $\frac{\partial^2 f}{\partial x^2}$, $\frac{\partial^2 f}{\partial^2 y}$, $\frac{\partial^2 f}{\partial x . \partial y}$ at the double point in question. If these three expressions are not all simultaneously zero, then the double point is a *node*, a *cusp in general*, or a *conjugate point* according as

$$\left(\frac{\partial^2 f}{\partial x . \partial y}\right)^2 - \frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2}$$

is positive, zero or negative.

Ex. 1. Examine the curve $a^2y^2=a^2x^2-4x^3$, for singular points.

Here $f(x, y)=a^2y^2-a^2x^2+4x^3=0$

(*i*) $\frac{\partial f}{\partial x}=-2a^2x+12x^2=0$; $\frac{\partial f}{\partial y}=2a^2y=0.$

$\therefore\ x=0,\ y=0$ } is a singular point. This is the only point which satisfies the three equations $f(x, y)=0$,

$$\frac{\partial f}{\partial x}=0 \text{ and } \frac{\partial f}{\partial y}=0$$

(*ii*) Again $\frac{\partial^2 f}{\partial x^2}=-2a^2+24x\ ;\ \frac{\partial^2 f}{\partial y^2}=2a^2\ ;\ \frac{\partial^2 f}{\partial x\,\partial y}=0.$

$\therefore$ at (0, 0), $\frac{\partial^2 f}{\partial y^2}=-2a^2\ ;\ \frac{\partial^2 f}{\partial y^2}=2a^2\ ;\ \frac{\partial^2 f}{\partial x\,\partial y}=0$

(*iii*) $\therefore\ \left(\frac{\partial^2 f}{\partial x.\partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}=+4a^4$, a positive quantity ;

the origin is a node ; the two real tangents (obtained by equating to zero, the lowest degree terms) at the origin are $y-x=0$ and $y+x=0$.

Ex. 2. Find the singular points of the curve

$$ay^2=(x-a)^2(x-b).$$

$$f(x, y)=ay^2-(x-a)^2(x-b)=0$$

$$\frac{\partial f}{\partial x}=-(x-a)^2-2(x-b)(x-a)=0\ ;\ \frac{\partial^f}{\partial y}=2ay=0.$$

$\therefore\ x=a,\ y=0$ } is a singular point, this being the only point which satisfies all the three equations $f(x, y)=0$, $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0.$

Also at $(a, 0)$, $\frac{\partial^2 f}{\partial x^2}=-2(x-a)-2(x-b)-2(x-a)$;

$$=-2(a-b)\ ;\ \frac{\partial^2 f}{\partial y^2}=2a\ ;\quad \frac{\partial^2 f}{\partial y.\partial x}=0$$

$$\therefore\ \mathrm{H}\equiv\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2-\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2}=4a(a-b)$$

Case (*i*) If $a<b$, H is negative.
$\therefore$ $(a, 0)$ is conjugate point.

Case (*ii*) If $a=b$, $H=0$.
$\therefore$ (a) is a cusp.

Case (*iii*) If $a>b$, H is positive.
$\therefore$ $(a, 0)$ is a node.

9. Form of a curve near the origin.

In order to *study the form of a curve* in the neighbourhood of a point, we must know.

(*i*) the tangent at the point ;

(*ii*) and the side of the tangent on which the curve lies ; whether the curve is above or below the tangent at the point.

Since we can always transfer the origin to the point under consideration, it is enough, to **find the form of the curve near the origin.**

(*a*) Now the tangent at the origin is easily found by equating the terms of the lowest degree to zero. Suppose **y=mx** is the *tangent* at the origin.

(*b*) To discover whether the curve is above or below the tangent, we want to express the given equation of the curve in the form

$$y=mx+bx^2+cx^3+\ldots\ldots \text{I} \qquad (curve)$$

For this purpose, we substitute for y in the equation of the curve and expand the terms and then equate the co-efficients of like powers of x in the resulting equation. We shall thus have sufficient number of conditions for determining the co-efficients b, c.....

Suppose the co-efficients b, c.....are determined and equation of the curve expressed in form I.

Let NQ ($=y_1$) and NT($=y_2$) be the ordinates of the curve and the tangent for the same abecissa ON$=x$ which is supposed to be small.

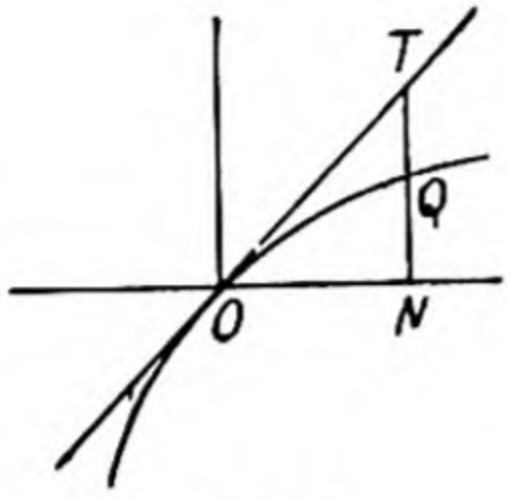


Fig. 1

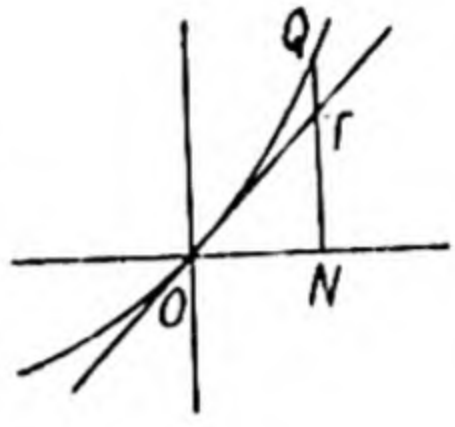


Fig. 2

Then TQ$=y_1-y_2=(mx+bx^2+cx^3+\ .\ )-mx$
$(bx^2+cx^3+\ldots\ldots$

When x is small, the sign of the right-hand side will be governed by bx^2 (*i. e.*, the term containing the lower power of x) which can be made greater than the sum of all the terms that follow it. Again, if y_1-y_2 or TQ be positive, the curve is above the tangent and if y_1-y_2 (or TQ) be negative, the curve is below the tangent. Hence the *curve is above or below the tangent according as b is positive* or *negative*, x^2 is always positive whether x is positive or negative.

If $b=0$ the sign of y_1-y_2 will depend on cx^3; if c be positive, the curve is above the tangent on the right of the y-axis (x-positive and below the tangent on the left of the y-axis (for x negative) and *vice versa* when c is negative. Thus the sign of b (or in its absence, the sign of c) determines whether the curve is *above* or *below* the tangent. If the origin be a double point, there will be two branches through the origin and two corresponding tangents and each branch is to be examined in relation to its tangent.

Note. Observe that the method adopted for expressing the equation of the curve in form I is similar to the *method* of *expansion* explained in connection with the determination of asymptotes where the expansion was in descending powers of x where in this investigation it is in ascending powers of x.

Ex. 1. Find the form near the origin of the curve

$$y^2-2x^2+xy+x^3-2y^3+x^4-2x^3y=0 \qquad (1)$$

(*i*) Origin is a double point on the curve, tangents at origin obtained by equating to zero the terms of the lowest degree are

$$-2x^2+xy+y^2=0, \quad i.\ e.,\ (y-x)(y+2x)=0$$

$\therefore$ $y=x$ and $y=-2x$ *are the tangents.*

(*ii*) To examine the curve with respect to the tangent $y=x$ in the neighbourhood of the origin we *want to express the equation of the curve in the form*

$$y=x+bx^2+cx^3+\ldots\ldots$$

Substituting for y in (1) we get

$$(x+bx^2+cx^3+\ldots)^2-2x^2+x(x+bx^2+cx^3+\ldots\ldots)$$
$$+x^3-2(x+bx^2+cx^3+\ldots)^3+x^4-2x^3(x+bx^2+cx^3+)\ldots=0$$

i. e., $x^3(2b+b+1-2)-x^4(\ \ \ldots)=0.$

Equating the co-efficients of x^3, x^4 .....to zero, we get $b=\frac{1}{3}$.....

$$\therefore \begin{cases} y=x\ldots\ldots & \textit{(tangent)} \\ y=x+\frac{1}{3}x^2+cx^3+ & \ldots\textit{(curve in vicinity of origin)} \end{cases}$$

Since $b(=\frac{1}{3})$ is positive,

ordinate of curve > *ordinate of tangent,*

$\therefore$ curve lies above the tangent whether x is positive or x is negative, since x^2 does not change sign with x.

Again, to *examine the curve with respect to the tangent* $y=-2x$, want to express the equation of the curve in the form

$$y=-2x+b_1x^2+c_1x^3+\ldots\ldots$$

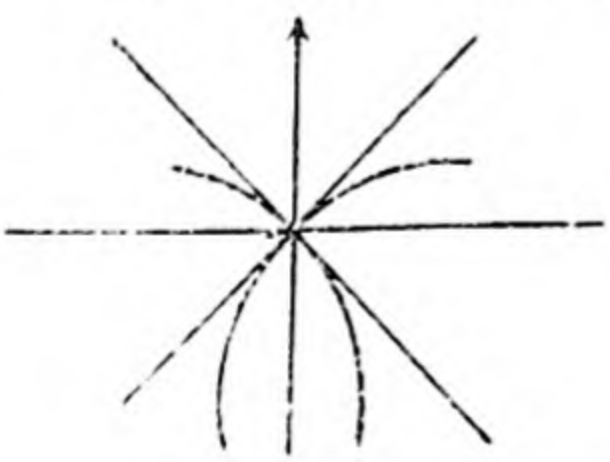
Substituting for y in the equation of the curve we get

$$(-2x+b_1x^2+\ldots)^2-2x^2+x(-2x^2+bx^2+c_1x^3+\ldots)$$
$$+x^3-2(-2x+b_1x^2+\ldots)^3+\ldots=0$$

i. e., $x^3(-4b_1+b_1+1+16)+x^4(\ldots\ldots)+\ldots=0$

Equating the co-efficient of x^3 to zero, we get $b_1=\frac{17}{3}$.

$$\therefore \quad \begin{cases} y=-2x & \textit{(tangent)} \\ \text{and} \quad y=-2x+\frac{17}{3}x^2+\ldots & \textit{(curve in the vicinity of origin)} \end{cases}$$



As before, the curve is above the tangent as $b_1=\frac{17}{3}$ is positive both when x is positive as well as when x is negative.

Ex. 2. Find the form near the origin of the curve

$$y^2(a^2+x^2)=x^2(a^2-x^2)$$

$$y=\pm x\left(1-\frac{x^2}{a^2}\right)^{\frac{1}{2}}\left(1+\frac{x^2}{a^2}\right)^{-\frac{1}{2}}$$

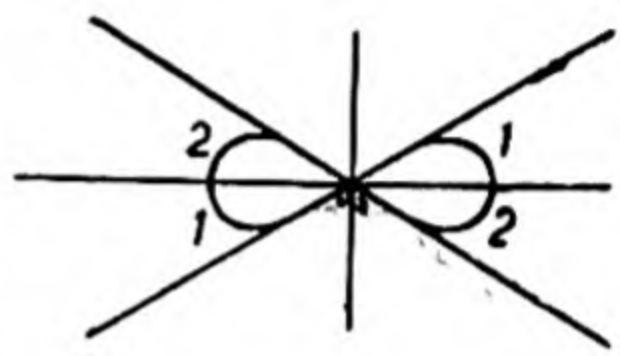
$$=\pm x\left(1-\tfrac{1}{2}\frac{x^2}{a^2}\ -\tfrac{1}{8}\frac{x^4}{a^4}-\ldots\right)\left(1-\tfrac{1}{2}\frac{x^2}{a^2}+\tfrac{3}{8}\ \frac{x^4}{a^4}\ \ldots\ \right)$$

$$=\pm x\left(1-\frac{x^2}{a^2}\ +\tfrac{1}{2}\ \frac{x^4}{a^4}\ \ldots..\right)$$

$\therefore$ The two branches of the curve are

(*i*) $y=x-\dfrac{x^3}{a^2}\ +\tfrac{1}{2}\ \dfrac{x^5}{a^4}\ldots..$

(*ii*) and $y=-x+\dfrac{x^2}{a^2}\ -\tfrac{1}{2}\dfrac{x^5}{a^4}\ \ldots$



Equation (*i*) gives the ordinate of the curve corresponding to the tangent $y=x$

Since the co-efficient of $x^3=-1/a^2$ the next term, is negative, the curve is below the tangent for x positive and above the tangent for x negative. Similarly, the curve is above the tangent $y=-x$ for branch 2 when x is positive and below the tangent when x is negative.

Classification of Cusps

10. A cusp is called a *cusp of the first species* (or a *ceratoid cusp*) when both branches of the curve lie on *opposite sides* of the common tangent. Such a cusp may be either *simple* or *double*; it is double when the branches extend on both sides of the point of contact and single when they do not so extend.

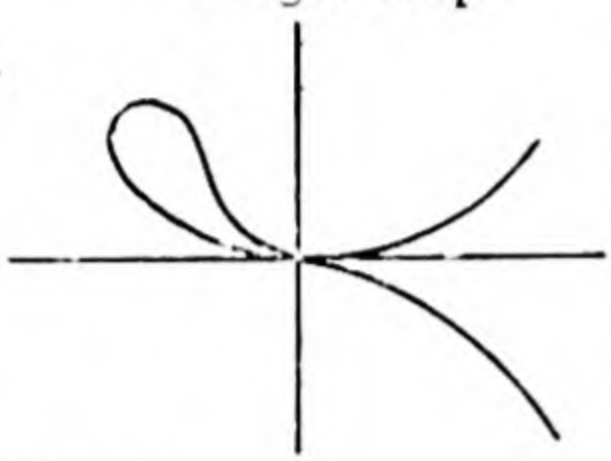
Cusp of first species
Single Cusp.

Cusp of first species
Double Cusp

A cusp is called a *cusp of second species* (or a *ramphoid cusp*) when both the branches lie on the *same side* of the common tangent. It may be either *single* or *double*.

Cusp of second species
Single Cusp

Cuspof second species
Double Cusp



A double cusp which is a combination of both species is called *a point of oscul-inflexion.* The cusp, in such cases, is also a point of inflexion.

Point of Oscul-inflexion, Double Cusp ; first species on the right ; second species on the left is shown in the figure.

Ex. 1. Find the forms near the origin of the curve

$$(y-x^2)^2=x^5$$

$\therefore \quad y=x^2\pm x^{\frac{5}{2}}$

(*i*) There are *two* branches

$$y=x^2(1+x^{\frac{1}{2}}) \text{ and } y=x^2(1-x^{\frac{1}{2}}).$$

(*ii*) Origin is a cusp, since $y^2=0$ (*x-axis*) are the two coincident tangents at the origin.

(*ii*) *x cannot be negative* as this makes $x^{\frac{5}{2}}$ imaginary and therefore, *y* imaginary Hence no position of the curve lies on the left of *y-axis*. Thus, *origin is a single cusp.*

(*iii*) When *x* is small, the sign of *x* depending on x^2, is positive *for both branches*, $\therefore$ both the *branches are above* the tangent (*x*-axis) $\therefore$ *origin is a cusp of the second kind* (*ramphoid cusp*).

(*v*) As *x* increases indefinitely from zero, the ordinate of the first branch continually increases, the second branch crosses *x*-axis, at $x=1$ the ordinate being negative after $x=1$, and descends indefinitely.

Ex. 2. Find the form near the origin of the curve $x^5+2a^2x^2y-a^3y^2=0$ (*a* positive).

(*i*) the curve passes through the origin ; the lowest degree terms equated to zero give *two coincident*

tangents $y=0$ (x-axis) at the origin. The origin is, therefore, *a cusp.*

(*ii*) the equation is a quadratic in y. Solving for y,

$y=x^2/a.(1\pm\sqrt{1+x/a})$.

(*iii*) (*a*) the origin is a double cusp because there are two values of y when x is positive as well as when x is negative, x being small ;

(*b*) when x is small and positive, the two values of x are of *opposits signs* $\therefore$ origin is *ceratoid cusp* on the right.

(*c*) when x is small and negative, both values of y are positive.

$\therefore$ origin is a ramphoid cusp on the left.

(*iv*) The curve consists of two branches

$y=x^2/a.(1+\sqrt{1+x/a})$ (*upper branch*)

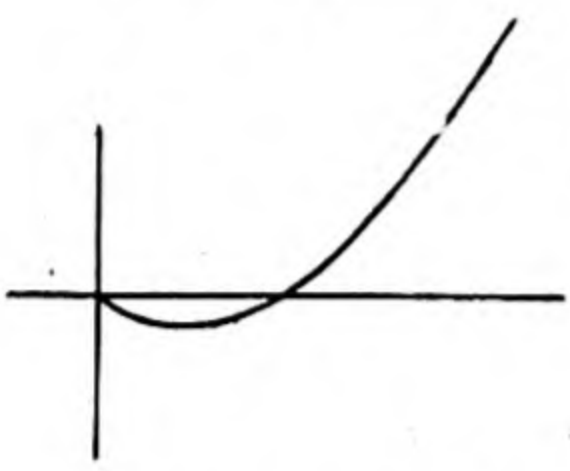
$y=x^2/a.(1-\sqrt{1+x/a})$ (*lower branch*)

For the upper branch, $\left(\frac{d^2y}{dx^2}\right)=\frac{4}{a}$ at the origin ; for the lower branch, $\left(\frac{d^2y}{dx^2}\right)=0$ and $\frac{d^2y}{dx^2}$ changes sign when x passes through zero. Therefore the lower branch has a *point of inflexion* at the origin.

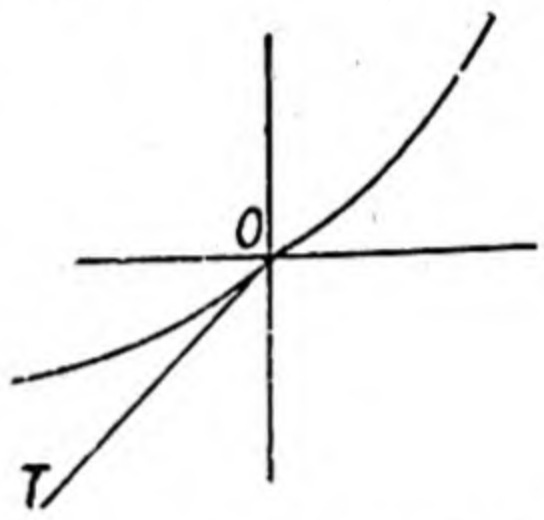
Thus, the origin is a double cusp, and also *a point of inflexion.* Hence it is a point of oscul-inflexion. There is no portion of the curve for x negative and *numerically* greater than a. (See last figure of Art. 10).

11. In order to study the curve $f(x, y)=0$ for singularities, we solve the equations $\frac{\partial f}{\partial x}=0$, $\frac{\partial f}{\partial y}=0$ simultaneously. Let (a, b) be one of the solutions. If (a, b) does not lie on the curve that is, if $f(a, b)$ is not equal to zero, we reject the point. However if $f(a, b)=0$, the point (a, b) is a singular point on the curve. To study further the nature of the singularity we shift the origin to the point (a, b) with the help of the relations $x=X+a$, $y=Y+b$, so that the equation of the curve now becomes $f(X+a, Y+b)=0$. Now we have to study the singularity at the origin for which a method has already been given.

12. Transcendental Singularities. A transcendental curve may have a *stop-point* at which it terminates abruptly. This is due to the discontinuity of the function at the point; or a *salient point* at which two branches of the curve meet and stop without having a common tangent due to a discontinuity in the derivative. Thus, $y=x \log x$ has a *stop point* at the origin since when $x \to 0$, $y \to 0$ but y is imaginary if x is negative.



Again $y=x/\left(1+e^{\frac{1}{x}}\right)$ has a *salient point* at the origin. Here $\frac{dy}{dx}$ has two different values at the origin. If $x \to +0$, $\frac{dy}{dx}=0$; if $x \to -0$, $\frac{dy}{dx}=1$, one branch touches the x-axis; the other touches OT which makes an angle of 45° with the x-axis. (Todhunter).



Ex. Examine for singularities the curve

$$x^3+2x^2+2xy-y^2+5x-2y=0.$$

Here $$\frac{\partial f}{\partial x}=3x^2+4x+2y+5=0 \qquad \ldots(1)$$

$$\frac{\partial f}{\partial y}=2x-2y-2=0 \qquad \ldots(2)$$

We have to solve these simultaneously.

From (2) putting $y=x-1$, we get

$$3x^2+4x+2(x-1)+5=0$$

or $$3x^2+6x+3=0 \text{ or } x^2+2x+1=0,$$

so that $x=-1$ and $y=-2$.

Substituting $(-1, -2)$ in the equations of the curve, we see that it lies on the curve and is, therefore, a singular point on the curve.

To examine its nature, let us shift origin to the point $(-1, -2)$ with the help of the relations $x=X-1$, $y=Y-2$.

The equation of the curve becomes

$$(X-1)^3+2(X-1)^2+2(X-1)(Y-2)-(Y-2)^2+5(X-1)-2(Y-2)=0$$

or $X^3-X^2+2XY-Y^2=0$ or $X^3-(X-Y)^2=0$,

which shows that (0, 0) may be a cusp on the curve, the tangent at origin being $Y-X=0$.

From the equation of the curve we get

$$Y-X=\pm X^{\frac{3}{2}} \qquad \text{or } Y=X\pm X^{\frac{3}{2}}$$

There are two branches of the curve

$$Y=X+X^{\frac{3}{2}} \text{ and } Y=X-X^{\frac{3}{2}}$$

Now X cannot be negative because if X is negative then Y is imaginary. Therefore no part of the curve lies to the left of the Y-axis. Thus origin is a single cusp.

Also the ordinate of one branch is greater than and ordinate of the other branch is less than the corresponding ordinate of the tangent. Hence the cusp is of the first species.

Thus at the point $(-1, -2)$ the given curve has a single cusp of the first species.

EXAMPLES XXXVI

1. Describe the points of inflexion of the curve $y=(x^3-x)/(3x^2+1)$ and show that the values of $\frac{dy}{dx}$ at those points are $\frac{1}{2}, -1, \frac{1}{2}$.

2. Show that the origin is the only point of inflexion on the curve $x^3-axy-b^2y=0$.

3. Show that the points of inflexion on the cubic $y=a^2x/(a^2+x^2)$ are given by $x=0$ and $x=\pm a\sqrt{3}$.

4. Determine the points of inflexion (if any) on the curve $y^2=a^3x/(a^2-x^2)$.

5. Find the points of inflexion on the curve $y^2=x(x+1)^2$.

6. Determine the points of the curve $y=2x^3-3x^2-12x+5$.

7. Find the points of inflexion of the curve $y^2=(x-2)^6(x-3)^5$.

8. Show that there is a point of inflexion at the origin on the curve $y=axy+by^2+cx^3$.

9. Show that there is a point of undulation at the origin on the curve $y=ax^4+bx^2y^2+cy^4$.

10. Show that the points of inflexion on the curve $x^2y=a^2(x-y)$ are given by $x=0$, $x=\pm\sqrt{3}a$.

11. Prove that the point $\left(-2, \frac{-2}{e^2}\right)$ is a point of inflexion on the curve $y=xe^x$.

12. Find the point of inflexion on the curve $y=e^{-\frac{1}{cx}}$.

13. Show that the curve $y=\sin x$ has no infinity of points of inflexion lying on the x-axis.

14. Show that the curve $y^2=ax^2+x^3$ has two points of inflexion when $a<0$ and none when $a>0$.

15. Show that the curve $y=ax^3+bx^2+cx+d$ has a point of inflexion when abscissa is one third of the sum of the roots of the equation $ax^3+bx^2+cx+d=0$, provided the roots are real.

16. Show that the ordinate of the point of inflexion on the curve $x=a-b\cos\theta$, $y=a\,\theta-b\sin\theta$ is $(a^2-b^2)/a$.

17. Find the point of inflexion on the curve $a^2=r^2\theta$.

18. Show that origin is the only point of inflexion on the curve $r^2=a^2\cos 2\theta$.

19. Examine the curve $y^3=2ax^2-x^3$ for singularities.

20. Find the multiple points (and their nature) on the curve $(x^2+y^2)^2=a^2(x^2-y^2)$.

21. Show that the origin is a node on the curve

$$x^3+y^3=3axy.$$

Find the equations of the tangents at the origin.

22. Show that origin is an isolated point on the curve

$$(x^2+y^2)^2=a^2x^2+b^2y^2.$$

23. Find the multiple points on the curve $ay^2-x^3+bx^2=0$ and discuss their nature.

24. Show that origin is a node, a cusp or a conjugate point on the curve $y^2=ax^2+bx^3$ according as a is positive, zero or negative.

25. Find the multiple points on the curve

$$y^6-3a^4y^2+2x^3-3ax^2+a^3=0.$$

and the value of $\frac{dy}{dx}$ at those points.

26. Find the double points on the curve

$$x^4-2y^3-3y^2-2x^2+1=0.$$

27. Find the double points on the curve

$$x^4+y^4-2x^2-2y^2+1=0$$

28. Find the double points on the curve

$$16(y^4-2y^3-2y^2)+(x^2-4)^2=0.$$

29. Show that origin is a triple point on the curve

$$x^4+y^4-2ay^3+2bx^2y=0.$$

Find the values of $\frac{dy}{dx}$ at the origin.

30. Show that there is a double cusp at the origin on the curve

$$x^5+16x^2y-64y^2=0.$$

[Tangents at the origin are $y^2=0$; hence there is a cusp at the origin. Further, $y=\frac{x^2}{8}(1\pm\sqrt{1+x}.)$ Near the origin x and y are small. When x is positive and small the two values of y are of opposite sign, so that the two branches lie on opposite sides of x-axis and the cusp is of the first species. When x is negative and small, the two values of y are of the same sign, the two branches therefore, lie on the same side of the x-axis and the cusp is of the second species. Hence there is a double cusp at the origin. It is easy to show that origin is a point of inflexion also.]

31. Find the multiple points and their nature for the curves

(i) $x^4-2x^2y+2xy^2+y^2=0$

(ii) $x^4-2x^2y-xy^2+y^3=0.$

32. Show that origin is a single cusp of the first species on the curve

$$y^2(2a-x)=x^3.$$

33. Determine the position and character of the singular points on the curve

$$x^4+y^3+2x^2+3y^2=0.$$

34. Show that the curve

$$x^4+2x^2y-xy^2-2x^3-2xy+y^2-x+2y+1=0$$

has a single cusp of the second kind at the point $(0,-1)$.

35. In the curve $a^3y^2-2abx^2y=x^5$ show that there is an oscul, inflexion at the origin.

36. Search for double points on the curve $(y-2)^2=x(x-1)^3$ and find the directions of the tangents there.

37. Show that at the point $(-1, -2)$ there is a cusp of the first species on the curve

$$x^3+2x^2+2xy-y^2+5x-2y=0.$$

38. Show that at each of the four points of intersection of the curve

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$$

with the axes there is a cusp of the first species.

39. Prove that if the curve $\dfrac{x}{f(t)}=\dfrac{y}{\phi(t)}=\dfrac{a}{\psi(t_1)}$

has a cusp at $t=t_1$, then $\dfrac{f'(t_1)}{f(t_1)}=\dfrac{\phi'(t_1)}{\phi(t_1)}=\dfrac{\psi'(t_1)}{\psi(t_1)}$.

40. Show that in the curve

$$x=a(t-\sin t)\ ;\ y=a(1-\cos t),$$

there is a cusp at $x=2na\pi$, $y=0$, n being an integer.

41. Show that a curve of the third degree cannot have more than one double point except when it degenerates into a conic and a straight line.

[For, if possible let it have two double points P and Q. Join P and Q. The line PQ then cuts the curve in four points, which is impossible.]

42. Find the double points and the value of $\frac{dy}{dx}$ there at for the curve

$$yx^2+y^2x+y^2-x^2+3x+3y=0.$$

Explain why the curve has two double points.

43. Find the nodal tangents and inflexional tangents to the curve

$$x^3+y^3+3(x^2+y^2)=3xy+1.$$

44. Show that in a cissoid the product of the perpendicular from the cusp on a tangent and the length of the tangent between the point of contact and the asymptote varies as the ordinate of the point of contact.

45. Determine the constants A, B, C so that the curve

$$v=\mathrm{A}(a^2-x^2)+\mathrm{B}\sin\frac{\pi x}{2}+\mathrm{C}\cos\frac{\pi x}{2a}$$

may cut the y-axis at an angle β at a distance k from the origin and that this point may be a point of inflexion.

46. Show that the point of inflexion of the curve $x=\phi(t)$ $y=\psi(t)$ are given by the equations.

$$\Delta\equiv\begin{vmatrix}\phi''(t) & \psi''(t)\\ \phi'(t) & \psi'(t)\end{vmatrix}=0$$

[The tangent at t_1 is $\{x-\phi(t_1)\}\,\psi'(t_1)-\{y-\psi(t_1)\}\phi'(t_1)=0$.

The perpendicular distance of a point (x, y) on the curve from the tangent is proportional to

$$\{\phi(t)-\phi(t_1)\}\psi'(t_1)-\{\psi(t)-\psi(t_1)\}\phi'(t_1)$$
$$=\tfrac{1}{2}(t-t_1)^2\begin{vmatrix}\phi''(t_1+\alpha) & \psi''(t_1+\alpha)\\ \phi'(t_1) & \psi'(t_1)\end{vmatrix}\quad\text{where } \alpha=\theta(t-t_1)\text{ and } 0<\theta<1.$$

Hence the curve crosses the tangent if Δ vanishes and changes sign at $t=t_1$. $\phi''(t)$ and $\psi''(t)$ are supposed to be continuous.]

CHAPTER XI

ENVELOPES

1. Family of curves. If (x, y) be the co-ordinates of a variable point in a plane, the equation $f(x, y)=0$ represents a curve. This equation usually contains certain constants that serve to fix the size, shape and position of the curve. For example, the equation $y=mx+a/m$ represents a straight line for every value of m ; its slope is m and its intercept on the y-axis is a/m. If we keep a fixed and give a series of values to m in succession, we get a series of straight lines. These straight lines may be said to constitute *a family*. The quantity m is constant for any straight line but is different for different straight lines which are *members of this family*. The quantity m which, by its variation, determines the family of straight lines is called a **parameter.**

In a similar manner, $(x-a)^2+y^2=r^2$ represents a circle whose radius is r and centre is $(a, 0)$. If r be constant and a varies continuously, the equation represents a *family of circles.*

The parameter of a family is usually denoted by α and to indicate this fact it is inserted in the functional symbol so that the family of curves having a *single parameter* is represented by

$$f(x, y, \alpha)=0.$$

Note. The student should observe that though the parameter α varies, its variation is independent of the variation of (x, y). Cases in which the family of curves contains **two parameters** connected by a relation will be considered later.

Def. *The* **envelope** *of a family of curves* $f(x, y, \alpha)=0$, *having* α *as the parameter of the family, is the curve (or group of lines) which touches every member of the family and which, at each point, is touched by some member of the family.*

EXAMPLES XXXVII

1. Write down the equation of the following and indicate the parameter in each case :—

(*i*) The family of straight lines

(*a*) passing through the origin.

(*b*) parallel to $y=mx$

(*c*) at a constant distance p from the origin.

(*ii*) The family of circles (*a*) having its centre on the x-axis and having a fixed radius equal to r, (*b*) which passes through the origin and whose centres are on the x-axis.

(*iii*) The family of parabolas whose foci are situated at different distances from the origin on the x-axis.

2. What systems are represented by these equations and what are the parameters ?

(*i*) $x^2+y^2=r^2$
(*ii*) $y^2=2ax-x^2$
(*iii*) $y=mx+\sqrt{1+m^2}$
(*iv*) $y=mx+\dfrac{a}{m}$
(*v*) $y=mx+\sqrt{a^2\pm b^2m^2}$
(*vi*) $x^2+y^2-2gx+c=0$
(*vii*) $y^2=4\alpha(x-\alpha)$
(*viii*) $\dfrac{x^2}{a^2}+\dfrac{y^2}{a^2-c^2}=1.$

3. Prove that the circle $x^2+y^2=p^2$ is the envelope of the family of straight lines $x\cos\alpha+y\sin\alpha=p$ for different values of α.

[*Hints.* (*i*) Show that $x\cos\alpha+y\sin\alpha=p$ touches the circle for all values of α and (*ii*) is touched at each point by one member of the family as the point of contact ($p\cos\alpha$, $p\sin\alpha$) is for all values of α a point on the circle.]

4. Show that the family of circles $(x-a)^2+y^2=a^2$ has no envelope, (a being the parameter).

[*Hints.* The y-axis touches each member of the family at the origin but each point on the y-axis is not a point of contact of some member of the family with the y axis.

This shows that *every family of curves has not necessarily an envelope.*]

2. To find the Envelope of f(x, y, a)=0.

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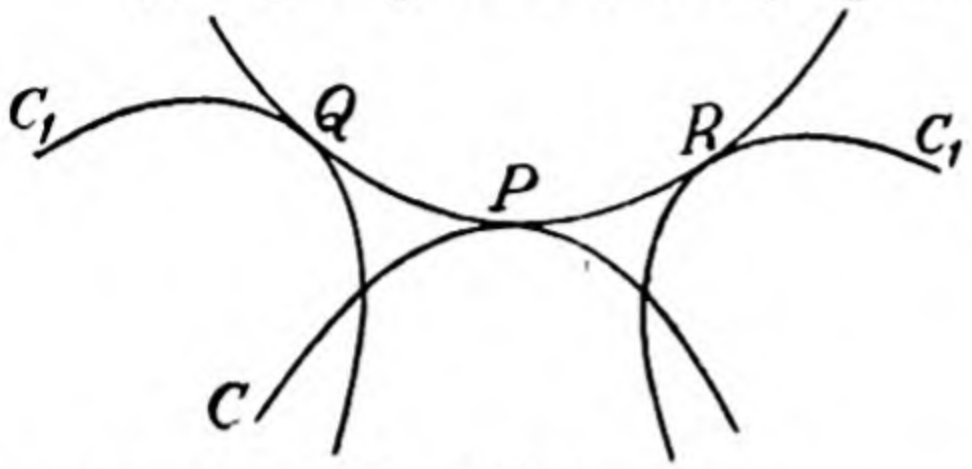


(*i*) By *definition*, every point P on the envelope is a point on one of the curves C of the family defined by a definite value of the parameter α.

$\therefore$ the co-ordinates (x, y) of a point on the envelope may be regarded as function of α. Thus, the *parametric equations of the envelope* may be written as

$$x=\phi(\alpha)\ ;\ y=\psi(\alpha) \qquad ..(1)$$

(*ii*) By definition the envelope touches each member of the family of curves $f(x, y, a)=0$ (2)

(*iii*) Now the *slope of the envelope* at $P(x, y)$ is

$$\frac{dy}{dx}=\frac{\frac{dy}{d\alpha}}{\frac{dx}{d\alpha}} \qquad (3)$$

The solope of the curve C at P is $\frac{dy}{dx}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}}$ (4)

Since (1) touches (2) the slopes (3) and (4) are equal for the same a.

$$\therefore \frac{\frac{dy}{d\alpha}}{\frac{dx}{d\alpha}}=-\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}};\ i.e.,\ \frac{\partial f}{dx}\frac{dx}{d\alpha}+\frac{\partial f}{\partial y}\frac{dy}{d\alpha}=0. \qquad (5)$$

(*iv*) Since *any point* P is on a curve C of the family the values of (x, y) given by (1) must satisfy (2). Hence for *all values of* a, $f\{\phi(\alpha), (\psi)\alpha, a\}=0$. (6)

$\therefore$ the total derivative, with respect to α, of the left hand side of (6) must be zero.

$$\therefore \frac{\partial f}{\partial x}\cdot\frac{dx}{d\alpha}+\frac{\partial f}{\partial y}\cdot\frac{dy}{d\alpha}+\frac{\partial f}{\partial \alpha}=0. \qquad (7)$$

Comparing (7) and (5), we get $\frac{\partial f}{\partial \alpha}=0$. (8)

Thus any point (x, y) on the envelope satisfies both the equations (2) and (8). Hence the equation of the envelope is obtained by eliminating α between (2) and (8). From the above we deduce the following **rule** *for finding the equation of the envelope involving a single parameter.*

Rule. (i) *Difference with respect to the parmater, regarding* x *and* y *as constants and obtain the equation* $\frac{\partial f}{\partial \alpha}=0$.

(*ii*) *Eliminate* α *between* $f(x, y, \alpha)=0$ and $\frac{\partial f}{\partial \alpha}=0$.

The resulting relation is the equation of the envelope.

This relation is evidently the condition that the equation $f(x, y, \alpha)=0$ in α may have two equal roots.

Note 1. The *parametric equations of the envelope* in form of equations (1) are obtained by solving for x, y in terms of α between the equations (2) and (8).

Note 2. We assume that f has continuous partial derivatives with respect to x, y and α.

Remarks. If the curves of the family contain double points (nodes or cusps) where $\frac{\partial f}{\partial x}=0$ and $\frac{\partial f}{\partial y}=0$, equation (5) is satisfied for all values of $\frac{dx}{d\alpha}$ and $\frac{dy}{d\alpha}$. In this case the result of eliminating α between $f=0$ and $\frac{\partial f}{\partial \alpha}=0$ may give, besides the envelope, other factors which when equated to zero will give loci which are not touched by members of the family of curves

$$f \equiv 9(y+c)^2-4x(x-3a)^2=0 \qquad (1)$$

$$\frac{\partial f}{\partial c}=18(y+c)=0. \qquad (2)$$

The result of elimination of c gives $x(x-3a)=0$.
$x=0$ is an envelope as it touches each member of the family for different values of c; $x-3a=0$ is not the envelope; it is a nodal locus passing through the nodes.

Ex. 1. Find the envelope of the family of circles $(x-a)^2+y^2=r^2$ obtained by varying α, r being fixed.

Here $f \equiv (x-\alpha)^2+y^2-r^2=0$ (1) $\dfrac{\partial f}{\partial \alpha}=2(x-\alpha)=0$ (2)

Eliminating α between (1) and (2), we have the equation of the *envelope*, $y^2-r^2=0$ or $y=\pm r$.

Thus the envelope is *two parallel lines.*

Ex. 2. Find the envelope of the family of concentric circles $x^2+y^2=r^2$ for different values of r.

$f \equiv x^2+y^2-r^2=0$ (1) $\qquad \dfrac{\partial f}{\partial r}=0+0-2r=0.$ (2)

Equation (2) does *not enable us to solve for the parameter r in terms of x and y.*

Hence there is no envelope, $r=0$ obtained from (2) gives only a particular member of the family and no envelope.

Ex. 3. Find the envelope of the family of circles

$$(x-\alpha)^2+y^2=a^2$$

Here $f \equiv x^2+y^2-2\alpha x=0$ (1) $\dfrac{\partial f}{\partial \alpha}=0+0-2x=0$ (2)

Equation (2) does not enable us to solve for a in terms of x or y or both. There is no envelope.

Note. From Ex. (2) and (3) we infer that a family of curves with a single parameter may not have an envelope, *i e*, there may not be any curve which touches each member and is touched at every point by a member of the family.

Ex. 4. Find the envelope of chords joining the extremities of conjugate diameters of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.

The equation of the chord of the ellipse joining the points whose eccentric angles are ϕ and ϕ' is

$$\frac{x}{a}\cos\frac{\phi+\phi'}{2}+\frac{y}{b}\sin\frac{\phi+\phi'}{2}=\cos\tfrac{1}{2}(\phi-\phi') \qquad (1)$$

If the points are the extremities of conjugate diameters, then $\phi'-\phi=\pi/2$, ϕ being the smaller of the two angles ϕ and ϕ';

(1) becomes

$$\frac{x}{a}\cos\frac{\phi+\phi+\pi/2}{2}+\frac{y}{b}\sin\frac{\phi+\phi+\pi/2}{2}=\frac{1}{\sqrt{2}}$$

or $$\frac{x}{a}\cos\left(\phi+\frac{\pi}{4}\right)+\frac{y}{b}\sin\left(\phi+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}. \qquad (2)$$

We are to find the envelope of (2) for different values of ϕ, the parameter.

Differentiating (2) with respect to ϕ, regarding x, y as constants, $$-\frac{x}{a}\sin\left(\phi+\frac{\pi}{4}\right)+\frac{y}{b}\cos\left(\phi+\frac{\pi}{4}\right)=0. \qquad (3)$$

To eliminate ϕ between (2) and (3), square and add (2) and (3) and we get $\frac{2x^2}{a^2}+\frac{2y^2}{b^2}=1$ as the envelope.

Ex. 5. Find the envelope of normals to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The equation of the normal to the ellipse at the point whose eccentric angle is ϕ is $ax/\cos\phi-by/\sin\phi=a^2-b^2$.

We have to *find the envelope of this line* for different values of the *parameter* ϕ.

Here $$f\equiv\frac{ax}{\cos\phi}-\frac{by}{\sin\phi}-(a^2-b^2)=0. \qquad (1)$$

$$\therefore\quad \frac{\partial f}{\partial\phi}\equiv ax\frac{\sin\phi}{\cos^2\phi}+by\frac{\cos\phi}{\sin^2\phi}=0. \qquad (2)$$

To eliminate ϕ, we solve for ϕ from (2) and substitute in (1).

From (2) $\tan^3\phi=-\frac{by}{ax}$ $\therefore\ \tan\phi=\frac{(-by)^{\frac{1}{3}}}{(ax)^{\frac{1}{3}}}=\frac{-(by)^{\frac{1}{3}}}{(ax)^{\frac{1}{3}}}$.

Hence $$\sin\phi=\frac{-(by)^{\frac{1}{3}}}{\sqrt{(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}}}$$

$$\text{and} \quad \cos\phi = \frac{(ax)^{\frac{1}{3}}}{\sqrt{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}}}.$$

Substituting these values in (I), we get

$$\frac{ax\sqrt{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}}}{(ax)^{\frac{1}{3}}} + \frac{by\sqrt{(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}}}}{(by)^{\frac{1}{3}}} = a^2 - b^2$$

i.e., $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$ which is the required envelope.

Ex. 6. Find the envelope of $A\lambda^2 + B\lambda + C = 0$, where A, B, C, are functions of x and y and λ is the parameter.

Differentiating, we get $2A\lambda + B = 0$, from which

$$\lambda = -\frac{B}{2A}.$$

Substituting this in the original equation we get

$$B^2 - 4AC = 0.$$

It may be noticed that the condition that

$$A\lambda^2 + B\lambda + C = 0$$

may have equal roots is also $B^2 - 4AC = 0$.

3. Envelope of family of curves involving *two parameters connected by an equation of condition.*

Let $f(x, y, a, b) = 0$ (1) be the family containing the parameters (a, b) which are connected by the relation

$$\phi(a, b) = 0. \qquad (2)$$

First Method. If possible, solve b in terms of a from (2).

Suppose $b = \psi(a)$. Substituting for (b) in (1) we get

$$f\{x, y, a, \psi(a)\} = 0.$$

This is a *family involving a single parameter.* Its envelope may, therefore, be obtained by the method already explained.

Second Method. Differentiate (1) and (2) with respect to a, regarding b as a function of a by virtue of equation (2);

we get

$$\frac{\partial f}{\partial a}+\frac{\partial f}{\partial b}\cdot\frac{db}{da}=0 \tag{3}$$

and $$\frac{\partial \phi}{\partial a}+\frac{\partial \phi}{\partial b}\cdot\frac{db}{da}=0 \tag{4}$$

Eliminate $\frac{db}{da}$ from (3) and (4), we get

$$\frac{\partial f}{\partial a}\cdot\frac{\partial \phi}{\partial b}-\frac{\partial f}{\partial b}\cdot\frac{\partial \phi}{\partial a}=0. \tag{5}$$

Eliminate a and b between (1), (2) and (5); the resulting equation is the equation of the envelope.

Ex. 1. Find the envelope of the family of lines whose intercepts on the axes have a constant sum k.

If a and b be the intercepts, then $\frac{x}{a}+\frac{y}{b}=1$ (1)

is the family of lines where a and b are *two parameters* subject to the relation $a+b=k$. (2)

First solution. Since $b=k-a$ from (2), (1) becomes

$$f\equiv x/a+y/(k-a)-1=0 \tag{3}$$

containing a single parameter a, k being an absolute constant.

$$\therefore\quad \frac{\partial f}{\partial a}\equiv -x/a^2-y/(k-a)^2\times -1=0 \tag{4}$$

i.e., $$(k-a)^2/a^2=y/x$$

$$\therefore\quad \frac{k-a}{a}=\frac{\sqrt{y}}{\sqrt{x}},$$ taking the positive sign for the radical

$$\therefore\quad a=\frac{\sqrt{x}.\,k}{\sqrt{x}+\sqrt{y}}.$$

Substituting in (3), we get, x, $\frac{\sqrt{y}}{\sqrt{x}}+y-(k-a)=0$

i.e., $$\sqrt{x}.\sqrt{y}+y-\frac{\sqrt{y}}{\sqrt{x}}\times\frac{\sqrt{x}.\,k}{\sqrt{\sqrt{x}+\sqrt{y}}}=0.$$

$$(\sqrt{x}+\sqrt{y})=\frac{k}{\sqrt{x}+\sqrt{y}}=0.$$

i.e., $\sqrt{x}+\sqrt{y}=\sqrt{k}$ (*Envelope*) which is a parabola touching the co-ordinate axes at distances k from the origin.

Second solution. We differentiate (1) and (2) with respect to a regarding b as a function of a :

$$\left.\begin{aligned} -\frac{x}{a^2}-\frac{y}{b^2}\frac{db}{da}=0 \\ 1+\frac{db}{da}=0 \end{aligned}\right\}$$

Eliminating $\frac{db}{da}$, $\frac{y}{b^2}=\frac{x}{a^2}$ ...(3)

Solving for a and b from (2) and (3), we get,

$$a=\frac{\sqrt{x}.k}{\sqrt{x}+\sqrt{y}},\ b=\frac{k\sqrt{y}}{\sqrt{x}+\sqrt{y}}.$$

Substituting for a and b in (1), we obtain $\sqrt{x}+\sqrt{y}=k$ as the *equation of the envelope.*

Ex. 2. Find the envelope of the co-axial ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad \ldots(1)$$

the sum of whose semi-axes is constant.

Here a and b are the parameters subject to the condition $a+b=c$ (an absolute constant). ...(2)

Differentiating (1) and (2) with respect to a regarding b as a function of a, we have

$$\frac{x^2}{a^3}+\frac{y^2}{b^3}\frac{db}{da}=0 \qquad \ldots(3)$$

and $$1+\frac{db}{da}=0 \qquad \ldots(4)$$

To find the envelope we eliminate a, b, $\frac{db}{da}$ from equation (1), (2), (3) and (4).

To do this, we first eliminate $\frac{db}{da}$ from (3) and (4) ; we get

$$\frac{x^2}{a^3}=\frac{y^2}{b^3}. \qquad \ldots(5)$$

From (2) and (5), solving for a and b

we get $a=\dfrac{cx^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}$; $b=\dfrac{cy^{\frac{2}{3}}}{x^{\frac{2}{3}}+y^{\frac{2}{3}}}$.

Substituting these values of a and b in (1), the equation of the envelope is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$.

4. *To prove that the envelope is, in general, the locus of the ultimate points of intersection of neighbouring curves.*

Let $$f(x, y, a)=0, \qquad (1)$$

$$f(x, y, a+\delta a)=0 \qquad (2)$$

be two neighbouring curves of the family. These curves intersect in one or more points. Let T be one of the points of intersection.

The co-ordinates of the points of intersection (such as T) satisfy equation (1) and (2) : therefore, they also satisfy their difference $f(x, y, a+\delta a)-f(x, y, a)=0$ (3)

Dividing (3) by δa, we obtain

$$\frac{f(x, y, a+\delta a)-f(x, y, a)}{\delta a}=0 \qquad (4)$$

Taking the limit when $\delta a \to 0$, (4) becomes $\dfrac{\delta f}{\delta a}=0$ (5)

The equation must be satisfied by the co-ordinates of the common points of intersection of (1) and (2) as these curves tend to coincidence with each other.

If we eliminate a between (1) and (5) we get the locus of the ultimate points of intersection for *all sets of neighbouring curves of the family*. But we have already seen (Art. 2) that this elimination leads to the equation of the envelope. Thus, we have established the theorem.

It may be remarked that by the ultimate point (or points) of intersection of a selected curve of the family with a neighbouring curve is understood the limiting position of the point (or points) of intersection as the neighbouring curve tends to coincidence with the selected curve.

***5. Modern Definition of Envelope.** An ordinary point on the curve $f(x, y, a)=0$ whose shortest distance from the curve $f(x, y, a+\delta a)=0$ is an infinitesimal of the second order at least is called a *characteristic point of the curve* (a). It can be shown that such points are ordinary points at which $f_a=0$ and are in general isolated. *The envelope of the family* $f(x, y, a)=0$ *is the locus of its isolated characteristic points.*

If there is an envelope, its points satisfy the equations

$$f=f_a=0.$$

The limit of a point of intersection of neighbouring curves, is an isolated characteristic point or a singular point.

The converse of this is not always true.

The locus of the ultimate intersection of neighbouring curves provided they intersect is in general the envelope, but it may be a locus of singular points. Conversely, the envelop is in general the locus of the ultimate intersections, but there may be an envelope when neighbouring curves do not intersect.

The envelope touches the curve (a) at all its isolated characteristic points that are ordinary poiuts of the envelope.

Conversely, a curve which touches just one member of the family at every point is in general the envelope or a part of it, but may be a locus of singular points.

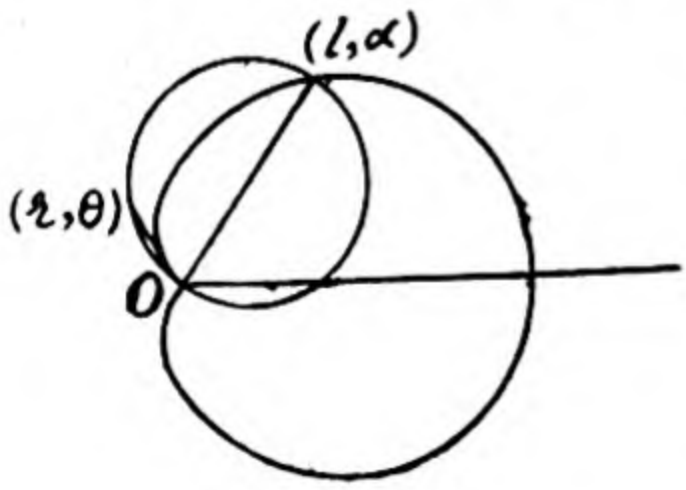
We took this as our definition.

The rigorous discussion of the theory of envelope is beyond the scope of this book and we will not therefore go beyond what is given above.

6. Worked out Examples.

Ex. 1. Find the envelope of the circle described on the radii vectors of the cardioid $r=a(1+\cos\theta)$ as diameters.

If (l, α) be the polar co-ordinates of any points on the cardioid, then $l=a(1+\cos\alpha)$ and the equation of the circle on l as diameter is $r=l\cos(\theta-\alpha)$



or (1) $r=a(1+\cos\alpha)\cos(\theta-\alpha)$.

Differentiating (1) w. r. t. parameter α,

$$0=-\sin\alpha\cos(\theta-\alpha)+(1+\cos\alpha)\sin(\theta-\alpha),$$

i. e., $$0=\sin(\theta-2\alpha)+\sin(\theta-\alpha)$$

or $$\sin(\theta-2\alpha)=-\sin(\theta-\alpha)$$

(2) $\therefore$, $\alpha=\frac{2}{3}\theta$ $\left[\because \theta-2\alpha=-(\theta-\alpha)\right]$

Substituting this value of α in (1), we get

$$r=a\left(1+\cos\frac{2}{3}\theta\right)\cos\frac{\theta}{3}$$

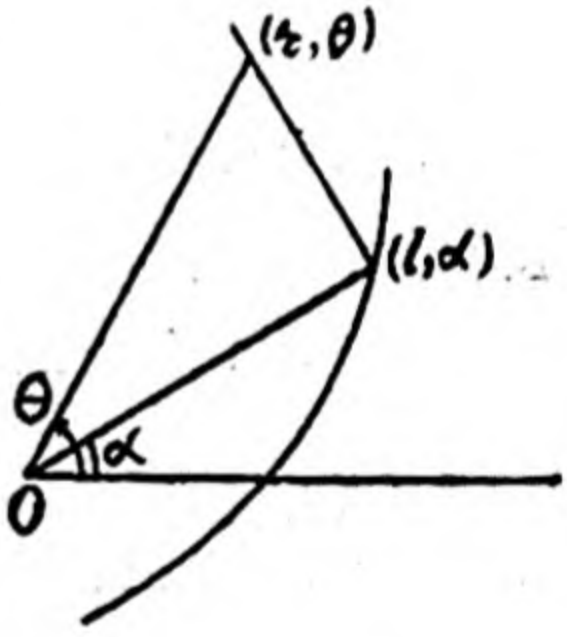
or $$r=2a\cos^3\frac{\theta}{3}$$

i. e., $$r^{\frac{1}{3}}=\left(2a\right)^{\frac{1}{3}}\cos\frac{\theta}{3},$$

as the reqd. equation of the envelope.

Ex. 2. Find the envelope of the straight lines at right angles to the radii vectors of the equiangular spiral

$$r=ae^{m\theta}.$$



Let (l, α) be the polar co-ordinates of any point on $r=ae^{m\theta}$. Then $l=ae^{m\alpha}$ the equation of the line through (l, α) perpendicular to the radius vector to (l, α) is

$$r\cos(\theta-\alpha)=l=ae^{m\alpha} \qquad \ldots\ldots(1)$$

Differentiating (1) w. r. t. parameter α we ge get

$$r\sin(\theta-\alpha)=ame^{m\alpha} \qquad \ldots\ldots(2)$$

We have to eliminate α between (1) and (2).
From (1) and (2) by division, we have

$$\tan(\theta-\alpha)=m$$
$$\theta-\alpha=\tan^{-1}m.$$
$$\therefore \quad \alpha=\theta-\tan^{-1}m.$$

Substituting this value of α in (1), we get as the required envelope

$$r\cos\{\theta-(\theta-\tan^{-1}m\}=ae^{m(\theta-\tan^{-1}m)}$$

i. e., $r\cos(\tan^{-1}m)=a^{-m\tan^{-1}m}.e^{m\theta}$

$$r=a.\sqrt{1+m^2}.e^{-m\tan^{-1}m}\times e^{m\theta}=Ae^{m\theta},$$

which is another equiangular spiral of the same type.

EXAMPLES XXXVIII

1. Find the envelope of

(*i*) $y=mx+a/m$ (*ii*) $y=mx+\sqrt{a^2m^2+b^2}$,

m being the parameter.

2. Find the envelope of $x^2/a^2+y^2/(c-a)^2=1$, a being the parameter.

3. Find the envelope of $y=x\tan\alpha-\frac{1}{2}g.\dfrac{x^2}{u^2\cos^2\alpha}$, α being the parameter.

4. Find the envelope of $y=mx+cm^n$, m being the parameter.

5. Find the envelope of $x\cos^n\theta+y\sin^n\theta=c$, θ being the parameter. Examine the case when $n=2$.

6. Show that the envelope of the family of straight lines $ax\sec\alpha-by\operatorname{cosec}\alpha=a^2-b^2$ is the curve

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}.$$

7. Find the curve of which the straight line

$$x\cos^3\theta+y\sin^3\theta=c$$

is always a tangent, whatever θ may be.

8. Find the envelope of the curves

$$\frac{a^2\cos\theta}{x}-\frac{b^2\sin\theta}{y}=\frac{c^2}{a},$$

θ being the parameter.

9. Find the envelope of the straight line

$$x/a+y/b=1$$

where (*i*) $a+b=c$ (*ii*) $a^2+b^2=c^2$ (*iii*) $ab=c^2$.

10. A straight line of given length slides between two given perpendicular straight lines. Find the envelope of the line.

11. Prove that the envelope of the family of ellipses

$$x^2/a^2+y^2/b^2=1.$$

(*i*) when $ab=c^2$ is $2xy=c^2$.

(*ii*) when $a+b=c$ is $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$.

12. Find the envelope of the parabola

$$\sqrt{x}/a+\sqrt{y}/b=1 \text{ where } a+b=c.$$

13. Find the envelope of normals to the parabola $y^2=4ax$.

14. Find the envelope of normals to the hyperbola

$$xy=c^2.$$

15. Show that the envelope of the family of the curves $A\lambda^3+3B\lambda^2+3c\lambda+D=0$, where λ is the parameter and A, B, C and D are functions of x and y is

$$(BC-AD)^2=4(BD-C^2)(AC-B^2).$$

16. Find the envelope of a circle whose centre lies on a given straight line and whose radius is proportional to the distance of its centre from a given point in that line.

17. Find the envelope of the circle which pass through the origin and have their centres on the hyperbola

$$x^2-y^2=a^2.$$

18. Show that the envelope of the family of circles described on the double ordinates of the parabola $y^2=4x$ as diameters is an equal parabola whose equation is

$$y^2=4a(x+a).$$

19. Show that the envelope of the family of the circles described on the central radius of ellipse $x^2/a^2+y^2/b^2=1$ as diameters is $(x^2+y^2)=a^2x^2+b^2y^2$.

20. Find the envelope of straight lines at right angles to the radii vectors of the following curves drawn through their extremities

(*i*) $r^n = a^n \cos n\theta$. (*ii*) $r = a + b\cos\theta$.

(*iii*) $r = ae^{\theta \cot \alpha}$.

21. A circle always passes through the origin and its centre lies on the parabola $y^2 = 4ax$; show that the envelope of the straight line joining the feet of these perpendiculars is

$$(x/a)^{\frac{2}{3}} + (y/b)^{\frac{2}{3}} = 1.$$

22. Find the envelope of the circle described on the radius vector of the following curves as diameters .

(*i*) $y^2 = 4ax$. (*ii*) $x^2/a^2 + y^2/b^2 = 1$. (*iii*) $r = a(1+\cos\theta)$.

(*iv*) $l/r = 1 + e\cos\theta$. (*v*) $r^n = a^n \cos n\theta$.

23. Prove that the value of $(1+y_1^2)^{\frac{2}{3}}/y_2$ for the envelope of the straight line $lx + my = 1$, where l and m are functions of the arbitrary parameter t, is

$$\frac{(l^2+m^2)^{\frac{3}{2}}\left(\frac{d^2l}{dt^2}\frac{dm}{dt} - \frac{dl}{dt}\frac{d^2m}{dt^2}\right)}{\left(l\frac{dm}{dt} - m\frac{dl}{dt}\right)^3}.$$

*CHAPTER XII

PEDALS INVERSES AND POLAR RECIPROCALS

1. Definition. If Y be the foot of the perpendicular OY on a variable tangent of a curve from a fixed point O, the locus of Y is called the **first positive pedal** (or more briefly, the **pedal**) of the curve with respect to O. Reciprocally, the original curve is called the *first negative pedal* of its pedal. Thus, the auxiliary circle of an ellipse or hyperbola being the locus of the foot of the perpendicular from a focus on a variable tangent, is *the first positive pedal* of the ellipse or hyperbola *with respect to the focus* ; similarly the pedal of a hyperbola *with respect to the focus* is the tangent at the vertex of the parabola.

Reciprocally, the first negative pedal of a circle is an ellipse or hyperbola according as the pole is inside or outside the circle ; the first negative pedal of a straight line is a parabola.

2. Given the Cartesian equation of a curve, show how to find the equation of its pedal.

Let $f(x, y)=0$ (i) be the equation of the curve and let $X\,\partial f/\partial x+Y\,\partial f/\partial y-(x\,\partial f/\partial x+y\,\partial f/\partial \lambda)=0$ (ii) be the tangent at (x, y).

Put (ii) in the form $X\cos\alpha+Y\sin\alpha-p=0$ (iii)

Comparing (ii) with (iii), we get

$$\frac{\cos\alpha}{\dfrac{\partial f}{\partial x}}=\frac{\sin\alpha}{\dfrac{\partial f}{\partial y}}=\frac{p}{x\dfrac{\partial f}{\partial y}+y\dfrac{\partial f}{\partial y}} \qquad (iv)$$

If x, y be *eliminated* between the *two equations* (iv) and equation (i), an equation between p and α [which is the condition that (iii) should touch (i)] is obtained. Since p and α are the polar co-ordinates of Y, this condition (relation between p and α) itself is the *polar equation of the first positive pedal.* To put the polar equation in the *customary form*, change p into r and α into θ. We thus

deduce the following **rule** *for finding the first positive pedal of* $(x, y)=0$.

(*a*) *Find the condition (in the form of a relation between* p *and* α*) that* $x\cos\alpha+y\sin\alpha-p=0$ *may touch the curve* $f(x, y)=0$.

(*b*) *In this relation put* $p=r$ *and* $\alpha=\theta$ *and we get the polar equation of the curve.*

Ex. Find the first positive pedal of the parabola $x^2=4ax$ with **respect to its vertex.**

Tangent to the parabola at (x', y') is

(*a*) $yy'=2a(x+x')$ *i.e.*, $2a.x-y'.y+2ax'=0$ (*i*)

(*b*) Comparing this with $x\cos\alpha+y\sin\alpha-p=0$ (*ii*)

we get $\cos\alpha/2a=\sin\alpha/(-y')=-p/2ax'$

$\therefore$ $y'=-2a\tan\alpha$ and $x'=-p\sec\alpha$,

(*c*) since $y'^2=4ax'$, we have, on substitution

$$4a^2\tan^2\alpha=4a\times-p\sec\alpha,$$

i.e., $a\tan^2\alpha=-p\sec\alpha$ [condition that (*ii*) may touch $y^2=4ax$].

Changing p into r and α into θ, we get

$a\tan^2\theta=-r\sec\theta$ (*polar equation of pedal*).

Again, putting $r=\sqrt{x^2+y^2}$, $\tan\theta=y/x$, we get

$$a.\ \frac{y^2}{x^2}=-\sqrt{x^2+y^2}\times\frac{\sqrt{x^2+y^2}}{x}$$

i.e., $ay^2+x(x^2+y^2)=0$ (*Cartesian equation of pedal.*)

Note. To find the pedal **with respect to the focus** $(a, 0)$, we transfer the origin to $(a, 0)$; then

$y^2=4a(x+a)$ *is the transformed equation of the parabola.*

The tangent at (x', y') is $yy'=2a(x+x'+2a)$

Comparing with $x\cos\alpha+y\sin\alpha-p=0$, we have

$\cos\alpha/2a=\sin\alpha/(-y')=-p/2a(x'+2a)$

$\therefore$ $y'=-2a\tan\alpha$ and $x'+2a=-p\sec\alpha$

since $y'^2=4a(x'+a)$, we have, on substitution

$$4a^2\tan^2\alpha=-4a(p\sec\alpha+a)$$

i.e., $$a\tan^2\theta=-(r\sec\theta+a)$$

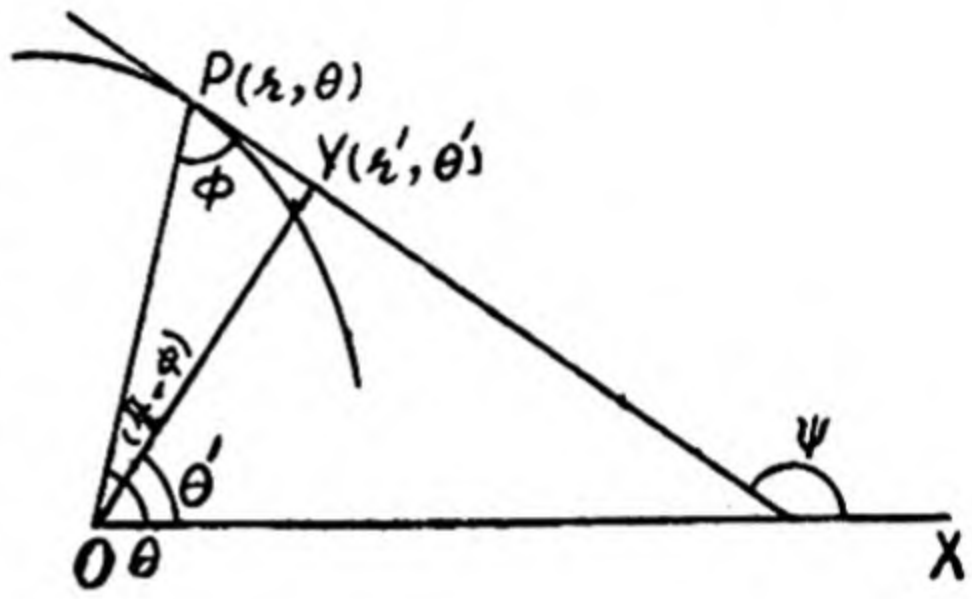
$\therefore a \sec^2\theta = -r \sec\theta$ (*polar equation of pedal referred to focus as pole*).

Hence the *cartesian equation*, (referred to focus as origin) *of the pedal* is

$x = -a$ (**which is the tangent at the vertex**).

3. Given the Polar equation of a curve: Show how to find the equation of the pedal curve.

Let $f(r, \theta)=0$ (i) be the equation of the curve. Let (r, θ) be the co-ordinates of P and (r', θ') those of Y, the foot of the perpendicular from O to the tangent at (r, θ). Then, from the figure



$$r' = r \sin\phi \qquad \ldots(ii)$$

where $\tan\phi = r\dfrac{d\theta}{dr}$ $\ldots(iii)$

also $\theta' = \theta - (\pi/2 - \phi)$ $\ldots(iv)$

If r, θ, ϕ be eliminated from (i), (ii), (iii) and (iv), the *resulting equation* in (r', θ), *will be the equation of the pedal.* The equation may then be put in the customary form by removing the dashes.

Note. Sometimes it is more convenient to replace equation (ii) by the equation $\dfrac{1}{r'^2} = \dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2$. The rest of the process is the same.

Ex. Find the first positive pedal of the curve $r^n = a^n \cos n\theta$.

Taking logarithm $n \log r = n \log a + \log \cos n\theta$.

(*a*) Differentiating $\dfrac{n}{r}\dfrac{dr}{d\theta} = -n \tan n\theta$.

$\therefore \cot\phi = -\tan n\theta \quad \left(\because \tan\phi = r\dfrac{d\theta}{dr}\right)$

$\phi = \pi/2 + n\theta$.

(*b*) Hence, $\theta' = \theta + \phi - \pi/2 = \theta + n\theta = \theta(n+1)$

i.e., $\theta = \theta'/(n-1)$.

(c) Now $r'=r\sin\phi=r\cos n\theta=a(\cos n\theta)^{\frac{1}{n}}.\cos n\theta$

$$=a(\cos n\theta)^{\frac{n+1}{n}}=a\left(\cos n.\frac{\theta'}{n+1}\right)^{\frac{n+1}{n}}$$

$$\therefore\ r'^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}.\cos\frac{n}{n+1}\theta'.$$

Removing the dashes, *the equation of the pedal* is

$$r^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\frac{n}{n+1}\theta,$$

a curve whose equation is of the same form as the original.

Note 1. Observe that the first positive pedal of

$r^n=a^n\cos n\theta$ is $r^{n_1}=a^{n_1}\cos n\,\theta$ where $n_1=n/(n+1)$.

Thus, putting $n=-2$ in equation (1), we get the equation of the first positive pedal of the curve $r^{-2}=a^{-2}\cot(-2)\theta$. (the *rectangular hyperbola* $r^2\cos 2\theta=a^2$) in the form

$$r^{\frac{-2}{-2+1}}=a^{\frac{-2}{-2+1}}\cos\left(\frac{-2}{-2+1}\right)$$

i.e., $r^2=a^2\cos 2\theta$ (*Lemniscate*).

Note 2. If the *first negative pedal* of $r^n=a^n\cos n\theta$ be $r^k=a^k\cos k\theta$, we have $n=k/(1+k)$ $\quad\therefore\ k=n/1(-n)$.

Hence, the first negative pedal of $r^n=a^n\cos n\theta$ is

$r^k=a^k$ oos $k\theta$ where $k=n/(1-n)$.

If perpendiculars are drawn to the tangent to the pedal from the origin we get a *new curve* passing through the feet of these perpendiculars, which is called the *second pedal of the curve* with respect to the origin.

In general, we may take the pedal of the pedal, positive or negative, with respect to the origin, *any number of times.* We then obtain a curve called the s^{th} *pedal, positive or negative.*

Ex. 1. Find the s^{th} *positive pedal* of $r^n=a^n\cos n\theta$.

The first positive pedal is $r^{n_1}=a^{n_1}\cos n_1\theta$ where

$$n_1=n/(n+1).$$

The second positive pedal is $r^{n_2}=a^{n_2}\cos n_2\theta$ where

$$n_2=n_1/(n_1+1)=\frac{n/(n+1)}{n/(n+1)+1}=n/(1+2n).$$

The third positive pedal is $r^{n_5}=a^{n_3}\cos n_3\theta$ where

$$n_3=n_2/(n_2+1)=\frac{n/(2n+1)}{n/(2n+1)+1}=n/(1+3n).$$

Proceeding in this manner, the s^{th} positive pedal is

$$r^{n_s}=a^{n_s}\cos n_s\theta \text{ where } n_s=n/(1+sn).$$

Ex. 2. Find the s^{th} *negative pedal of* $r^n=a^n\cos n\theta$.

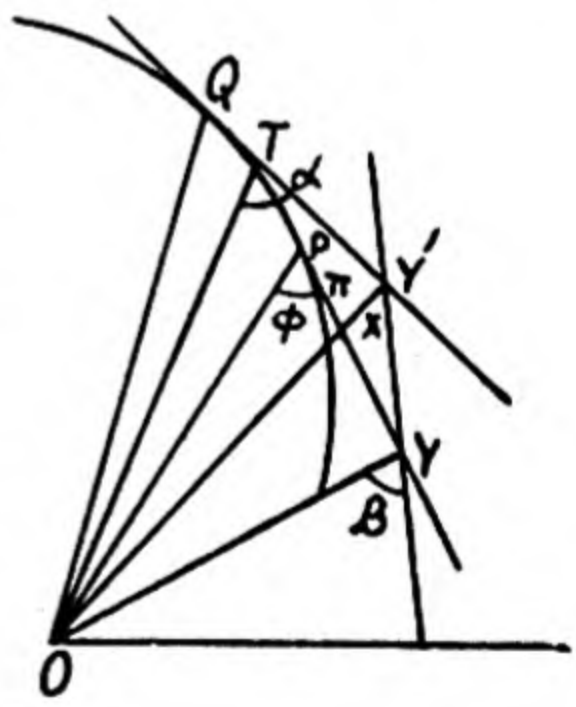
Let the s^{th} negative pedal of $r^p=a^p\cos p\theta$.

Then the s^{th} positive pedal of $r^p=a^p\cos p\theta$ is the original curve $r^n=a^n\cos n\theta$.

$\therefore\quad n=p/(1+sp)$, whence $p=n/(1-sn)$

$\therefore$ the sth negative pedal of $r^n=a^n\cos n\theta$ is $r^p=a^p\cos p\theta$, where $p=n/(1-sn)$.

4. Let OY, OY′, be the perpendiculars from the origin O on the tangents at two neighbouring points P and Q on the given curve and let T be the intersection of these tangents; join YY′.



Now $\angle OYT=\angle OY'T=$right angle.

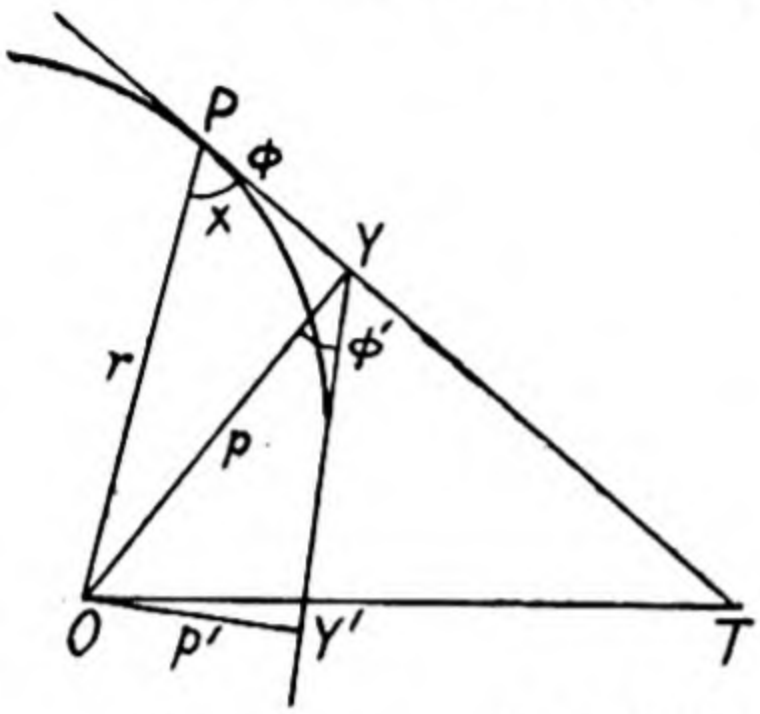
$\therefore$ the quadrilateral OYY′T is inscribable in a circle.

Hence, $\angle OTY'=\angle OYZ$ $\qquad(i)$

Therefore, in the limit when P tends to coincidence with Q, T also coincides with P and Y′Y becomes the tangent at Y to the pedal curve (locus of Y) and

$$\lim_{Q\to P}\angle OTY'=\lim_{Q\to P}\angle OYZ \text{ from } (i)$$

But the *limit of the left hand side* $= \angle \phi$, between the tangent and radius vector at any point P of the curve and the *limit of the right-hand side* $= \angle \phi'$ between the tangent and radius vector at *the corresponding point* Y of the pedal.



Hence, *the curve* and *its first positive pedal have the same radial angle* ϕ *at corresponding points.*

Corollary. Let OY′ be the perpendicular from O on YY′, the tangent at Y to the pedal curve; then calling $OY' = p'$ we have $\sin \phi = p/r$ from $\triangle$ OPY, and $\sin \phi' = p'/p$ from $\triangle$OYY′.

But we have just proved that $\phi = \phi'$ $\therefore$ $p'/p = p/r$.

Hence p', *the length of perpendicular on the tangent at any point of the pedal of a given curve is given by*

$$\mathbf{p' = p^2/r}.$$

This relation determines at once the pedal equation of the pedal of a given curve whose pedal equation $f(r, p) = 0$ is given. We simply change p into r and r into p^2/r in $f(r, p) = 0$.

Further, TR $\sin \delta\psi$ = RY′ (from the figure.)

In the limit $PY = dp/d\psi$.

Thus the projection of the radius vector to any point on a curve on the tangent at that point $= dp/d\psi$.

5. Pedals regarded as envelopes.

To show that (*i*) the first positive pedal of a given curve is the envelope of circles described on radii vectors as diameters.

(*ii*) that the first negative pedal is the envelope of a straight line drawn through any point of the curve and at right angles to the radius vector to the point.

(*i*) Draw a circle on OT as diameter (Fig. Art. 4). This circle passes through both Y and Y′, since $\angle Y = \angle Y'$.

But Y and Y′ are ultimately coincident points on the pedal as well as on the circle. $\therefore$ the line YY′ is ultimately a tangent to both the circle and the pedal. Hence the pedal

may be regarded as the envelope of circles drawn on radii vectors as diameters.

(*ii*) It is evident from the figure in Art. 4 that the first negative pedal is the envelope of a variable line PY which is drawn through Y perpendicular to the radius vector.

EXAMPLES XXXIX

1. Show that the first positive pedal of the ellipse $x^2/a^2+y^2/b^2=1$ with regard to the origin is $r^2=a^2\cos^2\theta+b^2\sin^2\theta$.

2. Show that the pedal of a parabola with respect to its vertex is the curve $x^2+y^2(x+a)=0$.

3. Show that the first positive pedal of a rectangular hyperbola $x^2-y^2=a^2$ with respect to its centre is $r^2=a^2\cos 2\theta$.

4. Show that the first positive pedal of the curve $x^n+y^n=c^n$ with regard to the origin is

$$(x^2+y^2)^{\frac{n}{n-1}}=a^{\frac{n}{n-1}}\left(x^{\frac{n}{n-1}}+y^{\frac{n}{n-1}}\right).$$

5. Show that the first positive pedal of the curve

$$\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$$

with regard to the origin is $r^{\frac{m}{m-1}}=(a\cos\theta)^{\frac{m}{m-1}}+(b\sin\theta)^{\frac{m}{m-1}}$

Examine the case when $m=2$.

6. Find the first positive pedal of the curve $r^n=a^n\cos n\theta$ with respect to the pole and deduce the kth positive pedal.

7. Find the first negative pedal of the curve $r^n=a^n\cos n\theta$.

8. Show that the first positive pedal of a circle with respect to any point is of the form $r=a+b\cos\theta$.

Examine the case when the point is on the circumference.

9. Show that the first positive pedal may be obtained by writing r instead of p and r^2/p instead of r in the pedal equation of the original curve.

Show that the first positive pedal of the curve

$$p^2=Ar^2+B \text{ is } r^2=Ar^4/p^2+B.$$

10. Show that the nth pedal of the curve $f(p, r)=0$ is

$$f(r^n/p^{n-1}, r^{n+1}/p^n)=0.$$

Find in this manner the pedal equation of the nth positive pedal of the curve $r^m=a^m \cos m\theta$.

11. Show that the negative pedal of the parabola $y^2=4ax$ with respect to the vertex is the curve $27ay^2=(x-4a)^3$.

Inverse Curves

6. Definition. Let O be a fixed point and P any point on the curve C. Take a point Q on OP or OP produced such that OP.OQ=constant=K^2. Then, the locus of Q (the curve C′, suppose) as P moves on the curve C is called the **inverse** *of the curve C with respect to the origin O.*

C and C′ are *inverse curves,* each being the inverse curve of the other; P and Q are *inverse points*, each being the inverse of the other. The process by which we pass from C to C′ (or from C′ to C) is called **inversion.** O being the *centre of inversion* and K is called the *constant of inversion.* C and C′ are sometimes said to be *inverse with respect to a circle of a radius K and centre O.*

7. Polar equation of the inverse.

Let $f(r, \theta)$ be the polar equation of the locus of P. Let (r, θ) be the co-ordinates of P and (r', θ) those of Q. The θ co ordinates of the two *corresponding* points P and Q must be the same since P and Q lie on the same radius vector through O

Now $rr'=K^2$ (*definition*)

$\therefore$ $r=K^2/r'$; substituting for r in $f(r, \theta)$, we get, $f(K^2/r', \theta)=0$, a relation between any r' and corrseponding Q, the polar co-ordinates of Q. This, therefore, represents the locus of Q. We put in the customary form by removing the dash from r', and obtain $f(K^2r, \theta)=0$ (*polar equation inverse of* $f(r, \theta)=0$.

The *equation of the inverse curve is, thus obtained by writing* K^2/r *for every* r *in the polar equation of the given curve.*

For example, the inverse of the curve $r^m=a^m \cos m\theta$ with respect to a circle whose radius is K and centre is pole is

$(K^2/r)^m=a^m \cos m\theta$, *i.e.*, $r^m \cos m\theta=(K^2/a)^m$.

Again, the *inverse of the parabola* $l/r=1+\cos\theta$ with regard to a circle of radius K and whose centre is the focus is the *cardioid* $r=a'(1+\cos\theta)$ where $a'=K^2/l$.

8. Cartesian equation of the inverse.

Let origin be the centre and K the radius of inversion; let the co-ordinates of a point P be (x, y) and those of its inverse be (X, Y). Then drawing perpendiculars PM and QN on the x-axis, we observe from similar triangles OPM, OQN, that $x/X=y/Y=OP/OQ=OP.OQ/OQ^2=K^2/(X^2+Y^2)$

$$\therefore\quad x=K^2X/(X^2+Y^2)\ ;\ y=K^2Y/(X^2+Y^2).$$

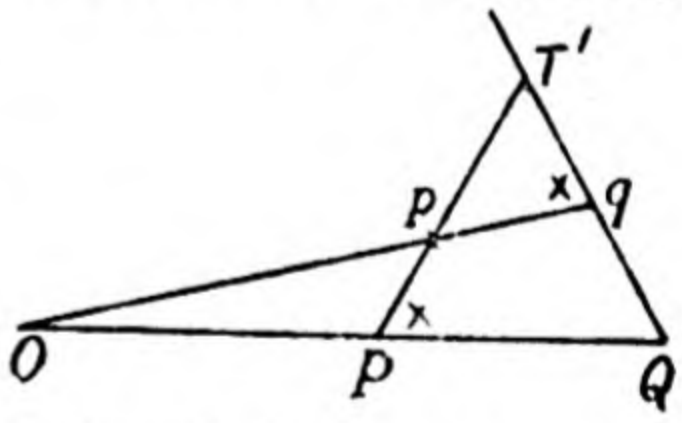
If $f(x, y)=0$ be the equation of the locus of P, then X, Y, will satisfy $f\left(\frac{K^2X}{X^2+Y^2}, \frac{K^2Y}{X^2+Y^2}\right)=0$.

This is, then, the equation of the locus of Q, the curve inverse to C. Denoting the current co-ordinates by (x, y), we get by changing the capital letters into small letters the equation in the customary form.

$$f\left(\frac{K^2x}{x^2+y^2}\ \ \frac{K^2y}{x^2+y^2}\right)=0$$ *(equation to the inverse of* $f(x, y)=0$.

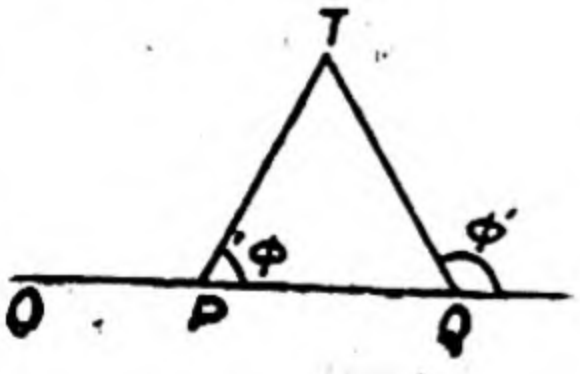
Note. The equation of the inverse curve is, therefore, obtained by writing $K^2x/(x^2+y^2)$ for every x and $K^2y/(x^2+y^2)$ for every y in the equation of the original curve.

9. Let P, p be two neighbouring points on a curve and Q, q; the corresponding points on the inverse curve, Then since $OP.OQ=K^2=Op.Oq$, the points P, Q, q, p are concyclic.



$$\therefore\quad \angle T'PQ=\angle T'qO \qquad ..(i)$$

In the limit when p tends to coincidence with P and q with Q, the chords Pp, Qq ultimately become tangents at corresponding points of the curve and its inverse and



$$\text{Lt}_{p\to P}\ \angle TPO=\text{Lt}_{q=Q}\ \angle T'qO \text{ from } (i)$$

But the *limit of the left-hand side* =radial angle ϕ at P and the *limit of the right-hand side* $=\pi-\phi'$, where ϕ' is the radial angle at Q

This shows that *the tangents at corresponding points P and Q form an isosceles triangle with the common radius vector passing through these points.*

If p and p' be the perpendiculars from the origin O on the tangents PT and QT and OP$=r$, OQ$=r'$, then

$$\frac{p}{p'}=\frac{r}{r'}. \quad \text{Hence } p=\frac{k^2p'}{r'^2}.$$

This relation yields at once the pedal equation of the inverse of the curve $f(r, p)=0$; we have simply to replace p by $\frac{k^2p}{r^2}$ and r by $\frac{k^2}{r}$ in $f(r, p)=0$.

Ex. 1. Find the inverse of the straight line $Ax+By+C=0$ with respect to a circle of radius K.

Putting $\frac{K^2x}{x^2+y^2}$ for x and $\frac{K^2y}{x^2+y^2}$ for y, we get

$$A.\frac{K^2x}{x^2+y^2}+B.\frac{K^2y}{x^2+y^2}+C=0$$

i.e., $Cx^2+Cy^2+AK^2x+BK^2y=0$ ...(*i*)

This is the equation of a circle which passes through the origin.

Special case. If C$=0$, (*i*) becomes $Ax+By=0$ which is identical with the given straight line.

The inverse of a straight line is the line itself or a circle according as it does or does not pass through the centre of inversion, provided that the slope of the line is not $\pm\sqrt{-1}$.

Ex. 2. Find the inverse of the circle

$$x^2+y^2+ax+by+c=0$$

with respect to a circle of radius k and centre at the origin.

Substituting $\frac{k^2x}{x^2+y^2}$ for x and $\frac{k^2y}{x^2+y^2}$ for y, we get the equation as

$$\left(\frac{k^2x}{x^2+y^2}\right)^2+\left(\frac{k^2y}{x^2+y^2}\right)^2+a.\frac{k^2x}{x^2+y^2}+b.\frac{k^2y}{x^2+y^2}+c=0.$$

$k^4x^2+k^4y^2+ak^2x(x^2+y^2)+bk^2y(x^2+y^2)+c(x^2+y^2)^2=0,$

i.e., $k^4+ak^2x+bk^2y+c(x^2+y^2)=0,$

i.e., $cx^2+cy^2+ak^2x+bk^2y+k^4=0$, which is a circle.

Special Case. If $c=0$, the equation of the inverse becomes $ax+by+k^2=0$ which is a straight line.

Thus, the *inverse of a circle is, in general a circle ; but the inverse of a circle which passes through the origin is a straight line.*

EXAMPLES XL

1. Show that the inverse of the curve $a(x+y)=xy$ with regard to the origin is $a(x^2+y^2)(x+y)=k^2xy$.

2. Show that the inverse of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with regard to the origin is $(x^2+y^2)^2=k^4\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$.

3. Show that the inverse of the curve $x^2-y^2=a^2$ with respect to the origin is $a^2(x^2+y^2)^2=k^4(x^2-y^2)$.

4. Prove that the inverse of the curve $r=ae^{\theta \cot \alpha}$ with regard to the pole is a curve of the same species.

5. Show that the inverse of the parabola $\frac{2a}{r}=1+\cos\theta$ with respect to the focus is $2ar=k^2(1+\cos\theta)$.

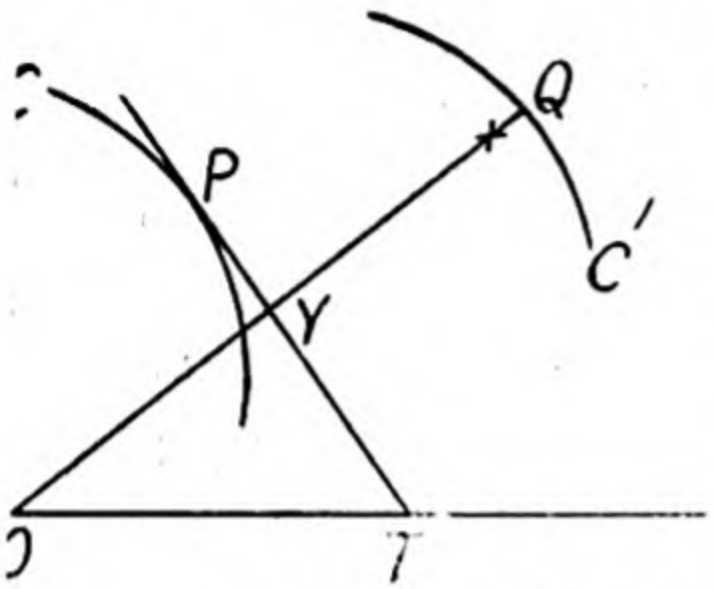
6. Show that the inverse of a curve $f(p, r)=0$ with regard to the pole is $f\left(\frac{k^2p}{r^2}, k^2/r\right)=0$.

7. Prove that by inversion, or otherwise, that the curves $r=a(1+\cos\theta)$ and $r=b(1-\cos\theta)$ cut each other at right angles.

8. Show by inversion that the circles $5(x^2+y^2)=3x+4y$ and $5(x^2+y^2)=4x-3y$ cut orthogonally.

Polar Reciprocals

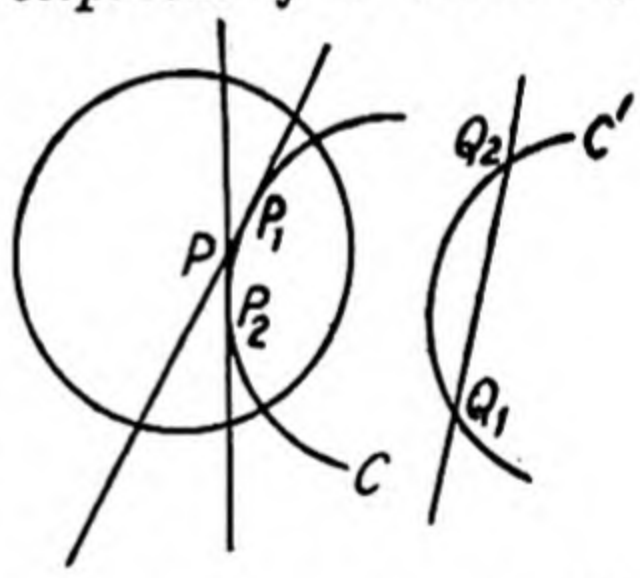
10. Definition. The inverse of the pedal of a curve with respect to the same pole is called the **polar reciprocal** of the curve.



If OY be drawn perpendicular from the pole on the tangent and if Q be taken on OY (or OY produced) such that OY. $OQ=k^2$, (constant), then the curve C′, the locus of Q, being the inverse of the locus of Y, the pedal of C, is the *polar reciprocal* of the curve C with regard to the circle of radius k and centre O.

Q is evidently the pole of the tangent PY with respect to this circle ; hence *the curve C′, the polar reciprocal of C is the locus of the tangents to C·*

11. *To show that if C′ be the polar reciprocal of a curve C, then C is the polar reciprocal of C′.*



We know that the *point of intersection of any two lines is the pole of the line joining their poles.* Hence P is the pole of the line Q_1Q_2 where Q_1 and Q_2 are the poles of the tangent to the curve C at P_1 and P_2. Now, let Q_1 tend to coincidence with Q_2, the line Q_1Q_2 will ultimately become a tangent to C′ at Q_2 ; but as Q_1 approaches to coincidence with Q_2, the two tangents to the curve C at P_1 and P_2 will become the tangent at P_2 and the limiting position of P is evidently P_2. Hence P_2 is the pole of the tangent to C′ at Q_2. Thus, C *is the locus of the poles of tangents to C′.*

It is easy to show that the polar reciprocal of a curve may also be regarded as the envelope of the polars of the points with respect to the circle.

Ex. 1. Find the polar reciprocal of the parabola $y^2=4ax$, with respect to the circle $x^2+y^2=4$.

First method. Regarding the polar reciprocal as the inverse of pedal.

(*i*) We find the condition that $x\cos\alpha+y\sin\alpha-p=$ touches $y^2=4ax$; we deduce that the *equation of the pedal* o parabola is $ay^2+x(x^2+y^2)=0$.

(*ii*) To find the inverse of the pedal $ay^2+x(x^2+y^2)=0$, w substitute $\frac{4x}{x^2+y^2}$ for x and $\frac{4y}{x^2+y^2}$ for y in this equation, sinc $k^2=4$. We thus get

$$a.\ \frac{16y^2}{(x^2+y^2)^2}+\frac{4x}{x^2+y^2}.\left(\frac{16x^2}{(x^2+y^2)^2}+\frac{16y^2}{(x^2+y^2)^2}\right)=0$$

i.e., $a.\ (x^2+y^2)16y^2+16\times 4x(x^2+y^2)=0$

i.e., $ay^2+4x=0$ or $y^2=-\frac{4}{a}\,x$ (*a parabola*).

The polar reciprocal is another parabola *turned to the left.*

Second method. We regard the polar reciprocal as th *locus of poles of tangents* to the parabola $y^2=4ax$.

Tangent at (x', y') on the parabola is $yy'=2a(x+x')$
i.e., $2ax-yy'+2ax'=0$. (a

If (α, β) be the poles of this tangent with respect to th circle $x^2+y^2=4$, we have $x\alpha+y\beta-4=0$ as *polar of* (α, β). (i

Identifying (a) and (b), we have $\frac{\alpha}{2a}=\frac{\beta}{-y'}=\frac{4}{-2ax'}$

whence $x'=-\frac{4}{\alpha}$ and $y'=-\frac{2a\beta}{\alpha}$.

Since (x', y') is on the parabola $y'^2=4ax'$.

Hence $4a^2\,\frac{\beta^2}{a^2}=4a.\left(-\frac{4}{\alpha}\right)$ *i.e.*, $\beta^2=-\frac{4}{a}\,\alpha$.

Changing β into y and α into x, we get $y^2=-\frac{4}{a}\,x$.

Third method. Let P $(at^2, 2at)$ be any point on the para bola $y^2=4ax$. The polar of P with respect to $x^2+y^2=4$ i $at^2x+2aty-4=0$. The envelope of this line is evidentl $ay^2+4x=0$, *i.e.*, $y^2=-\frac{4}{a}\,x$.

Ex. 2. Find the polar reciprocal of the curve

$$r^m=a^m\cos m\theta,$$

with respect to a circle of radius k.

(*i*) The pedal of $r^m = a^m \cos m\theta$ is

$$r^{\frac{m}{m+1}} = a^{\frac{m}{m+1}} \cos \frac{m}{m+1}\theta \qquad \text{(Ex. Page 274).}$$

(*ii*) Putting $\frac{k^2}{r}$ for r, we get $\dfrac{k^{\frac{2m}{m+1}}}{r^{\frac{m}{m+1}}} = a^{\frac{m}{m+1}} \cos \frac{m}{m+1}\theta.$

i.e., $r^{-\frac{m}{m+1}} \cos \frac{m}{m+1}\theta = \left(\frac{k^2}{a}\right)^{\frac{m}{m+1}}$, as the equation of the polar reciprocal.

EXAMPLES XLI

Find the polar reciprocals with respect to a circle of radius k and centre at the origin of the curves

1. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. **2.** $x^2+y^2=ax$. **3.** $r=a(1+\cos\theta)$.

4. $l/r=1+e\cos\theta$. **5.** $r^n=a^n\cos n\theta$. **6.** $x^m+y^m=c^m$.

7. Show that the polar reciprocal of the curve, $r=ae^{\theta \cot a}$ is a curve of the same species.

8. Prove that the polar reciprocal of the curve $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=1$ with respect to the circle described on the line joining the points $(\pm\sqrt{a^2-b^2}, 0)$ as diameter is $\frac{a^2}{x^2}+\frac{b^2}{y^2}=1$.

9. Show that the polar reciprocal of the curve $p=f(r)$ with respect to a circle of radius k and centre at the pole is

$$\frac{k^2}{r}=f\left(\frac{k^2}{p}\right).$$

10. Show that a node on a curve corresponds to a double tangent on the reciprocal curve and *vice versa*.

11. Show that a cusp on a curve corresponds to an inflexional tangent on the reciprocal curve and *vice versa*.

12. Show that the polar reciprocal of an equiangular spiral with respect to a rectangular hyperbola having its centre at the pole and touching the spiral is the curve itself.

CHAPTER XIII

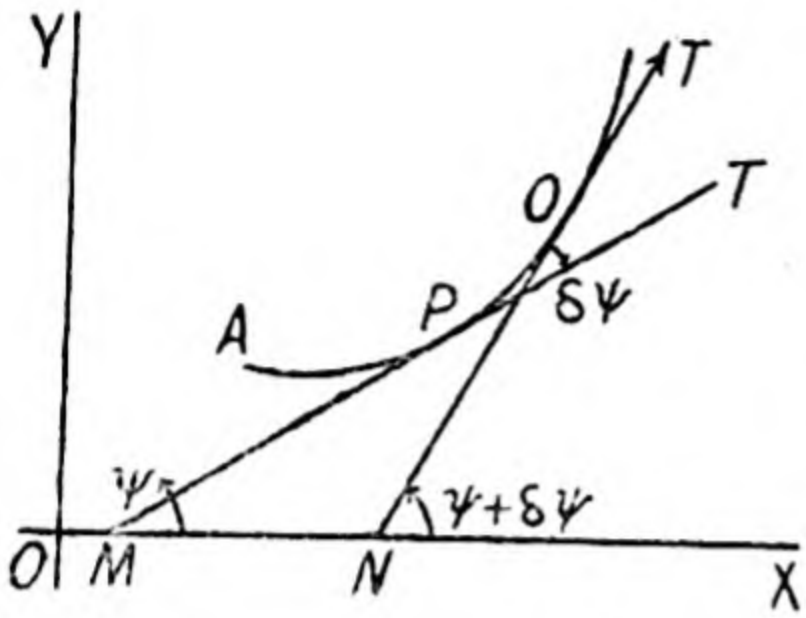
CURVATURE OF PLANE CURVES

1. So far we have considered the direction of the tangent at various points of a curve. We will now study the problem: 'How does the direction vary from point to point, *i.e.*, what is its rate at the point of contact?' This problem falls within the realm of what is called curvature. The notion of curvature was suggested to us by dynamical considerations. A curve may be regarded as the locus of a point or as the envelope of a straight line. It may also be regarded as generated simultaneously by a point P and a straight line PT. Then at any instant P is moving along PΓ and PT is rotating about P. The curvature expresses the relation between these two motions.

Let P move a distance δs along the curve and PT turn through an angle $\delta\psi$ in time δt. The curvature of the curve at P is defined to be the ratio of the angular velocity $\frac{d\psi}{dt}$ of PT to the velocity $\frac{ds}{dt}$ of P. If we think of P as describing the curve with unit velocity the curvature at P will be the angular velocity of PT. Thus if we denote the curvature at P by $\frac{1}{\rho}$, then $\frac{1}{\rho} = \frac{d\psi}{dt} \Big/ \frac{ds}{dt} = \frac{d\psi}{ds}$.

But this result, which is independent of t, goes to point out that there is nothing dynamical in the notion itself. The curvature of a curve can be defined without any reference to dynamics, and the theory of curvature can be developed from various independent stand-points which though logically distinct, yield identically the same formulas.

2. Let P and Q be two points on a curve which has a tangent at every point of the arc PQ. Let δs be the length of the arc PQ and $\delta\psi$ the angle between the tangents at P and Q.

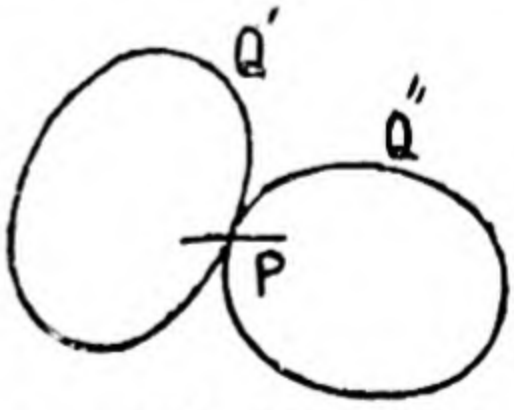


The angle $\delta\psi$, through which the tangent PT turns as the point of contact P moves along the arc from P to Q is defined to be the total or integral curvature of the arc PQ. $\delta\psi$ is also called the *angle of contingence* of the arc PQ; $\frac{\delta\psi}{\delta s}$ is called the *average or mean curvature* of the arc PQ.

And Lt $\frac{\delta\psi}{\delta s}$ when $Q \to P$ is defined as the curvature of the curve at P, provided such limit exists. Denoting this limit by $\frac{1}{\rho}$, we have $\frac{1}{\rho} = \frac{d\psi}{ds}$. Curvature thus measures the rate of the bending of the curve, *i.e.*, the rate at which the tangent PT is turning, with respect of the arc $AP = s$, as the point of contact P moves along the curve. A being a fixed point on the curve, ρ is called the *radius of curvature at* P, and

$$\rho = \frac{ds}{d\psi}.$$

If the curvature at P be zero, we may say that either no radius of curvature exists on the radius of curvature is infinite. The choice is perfectly at our will.



Sign of ρ. If two curves touch each other externally at P, the tangents rotate in opposite directions as the points of contact move from P to Q′ and Q″. One is therefore being described counter-clockwise and the other clock-wise.

Having regard to this point, we define the average curvature of the arc PQ as $\frac{\psi_Q - \psi_P}{s_Q - s_P}$ and the curvature at

$$P \text{ as } \operatorname*{Lt}_{Q\to P} \frac{\psi_Q - \psi_P}{s_Q - s_P}, \text{ if this limit exists.}$$

The radius of curvature, ρ, is the reciprocal of this limit.

The centre of curvature at P is a point C on the normal at P such that $PC=\rho$, due regard being had to the sign of ρ and the positive direction of the normal at P.

The circle of curvature at P is a circle having its centre at C and its radius $=\rho$.

Thus the circle of curvature has the same tangent, curvature and normal or concavity as the curve itself.

(This notation is due to Fowler).

Remarks. It may be remarked that the normal at P is drawn in the direction in which the curve turns its concavity. For example a normal to an ellipse at P is drawn inwards. This is the positive direction of the normal. It therefore follows that the curve is always concave to its centre of curvature,

It follows that (*i*) *a straight line has no curvature and* (*ii*) *the curvature of a circle at every point is equal to the reciprocal of its radius.*

If at P, $\frac{d\psi}{ds}=0$, then P is an *inflexional point* unless the curve is a st. line, for the rotation of the tangent is momentarily arrested and the curve has a *stationary tangent.* Here the direction of rotation is reversed after the stoppage.

If at P, $\frac{ds}{d\psi}=0$, P is a cusp, for the motion of the point along the line is momentarily arrested. In such a case, the direction of motion of the point is reversed after the stoppage.

EXAMPLES XLII

1. Show that for the catenary $s=c \tan \psi$, the radius of curvature is equal to the length on the normal terminated by the directrix.

GIRLS CO

2. Show that in the cycloid $s=4a \sin\psi$, the radius of curvature is double the normal terminated by the base.

3. Show that in the parabola $y^2=4ax$, $\rho=-\dfrac{2a}{\sin^3\psi}$.

4. Show that for the tractrix $s=a \log \operatorname{cosec}\psi$, the curvature varies as the normal.

5. Prove that $\rho=2a \sec^3\psi$ for the curve

$$s=a \log \cot\left(\frac{\pi}{4}-\frac{\psi}{2}\right)+a\frac{\sin\psi}{\cos^2\psi}.$$

6. Prove that for the curve $s=m(\sec^3\psi-1)$

$$\rho=3m \tan\psi \sec^3\psi.$$

Hence show that $3m\dfrac{dy}{dx}\cdot\dfrac{d^2y}{dx^2}=1$.

[*Hint*: Differentiate $\tan\psi=\dfrac{dy}{dx}$].

3. $\rho=\dfrac{ds}{d\psi}$ is not always the most convenient formula to use. This must, therefore, be transformed so as to suit each system of co-ordinates. And this can easily be translated into various forms. In what follows we assume the existence of all derivatives that occur.

(*a*) When x and y are given as functions of s.

Let $x=f(s)$; $y=\phi(s)$; $\cos\psi=\dfrac{dx}{ds}$, $\sin\psi=\dfrac{dy}{ds}$.

Differentiate each again and we have

$$-\sin\psi\frac{d\psi}{ds}=\frac{d^2x}{ds^2},\quad \cos\psi\frac{d\psi}{ds}=\frac{d^2y}{ds^2}.$$

$$\therefore\quad \frac{1}{\rho}=\frac{-\dfrac{d^2x}{ds^2}}{\dfrac{dy}{ds}}=\frac{\dfrac{d^2y}{ds^2}}{\dfrac{dx}{ds}} \qquad \ldots(1)$$

$$\text{whence}\quad \frac{1}{\rho^2}=\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2 \qquad \ldots(2)$$

(*b*) **Cartesian Form.** *To find the radius of curvature of the curve* $y=f(x)$.

From $\tan\psi=\frac{dy}{dx}=y'$, we have

$$\sec^2\psi\frac{d\psi}{ds}=\frac{d}{ds}\left(\frac{dy}{dx}\right)=\frac{d^2y}{dx^2}\cdot\frac{dx}{ds}=\cos\psi\,\frac{d^2y}{dx^2}$$

$$\therefore\quad \frac{ds}{d\psi}=\frac{\sec^3\psi}{\frac{d^2y}{dx^2}}\quad\text{or}\quad \rho=\frac{(1+y'^2)^{\frac{3}{2}}}{y''}\qquad\ldots\ldots(3)$$

Note. We make the convention that the radical $(1+y_1^2)^{\frac{3}{2}}$ in the numerator should be taken positively so that ρ will be positive or negative according as $\frac{d^2y}{dx^2}$ is positive or negative.

(*c*) *To find the radius of curvature of the curve* $f(x, y)=0$

$f_x+f_yy'=0$, *i, e.,* $y'=-\frac{f_x}{f_y}$.

$\therefore\quad y''=(f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2)/f_y^3.$

Substitution in (3) gives

$$\rho=\frac{(f_x^2+f_y^2)^{\frac{3}{2}}}{f_{xx}f_y^2-2f_{xy}f_xf_y+f_{yy}f_x^2}\ \text{(in magnitude)}\ldots\ldots(4)$$

Besides the existence of partial derivatives, we also *assume* $f_{xy}=f_{yx}$.

(*e*) **Parametric Form.** *To find the radius of curvature of the curve* $x=f(t)$, $y=\phi(t)$.

$$\frac{dy}{dx}=\frac{\phi'(t)}{f'(t)}$$

$$\therefore\ \frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{\phi'(t)}{f'(t)}\right]=\frac{d}{dt}\left[\frac{\phi'(t)}{f'(t)}\right]\frac{dt}{dx}=\frac{f'\phi''-\phi'f''}{f'^3}.$$

Substituting in (3), we get

$$\rho=\frac{(f'^2+\phi'^2)^{\frac{3}{2}}}{f'\phi''-\phi f''}=\frac{\left\{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right\}}{\frac{dx}{dt}\,\frac{d^2y}{dt^2}-\frac{dy}{dt}\,\frac{d^2x}{dt^2}}\qquad\ldots\ldots(5)$$

Ex. 1. Show that the radius of curvature at any point of the parabola $y^2=4ax$ is

$$\frac{2(x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}$$

We have, $y=2a^{\frac{1}{2}}x^{\frac{1}{2}}$; $\frac{dy}{dx}=\frac{a^{\frac{1}{2}}}{x^{\frac{1}{2}}}$; $\frac{d^2y}{dx^2}=-\frac{a^{\frac{1}{2}}}{2x^{\frac{3}{2}}}$

$$\therefore \quad \rho=\frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}=\frac{\left(1+\frac{a}{x}\right)^{\frac{3}{2}}}{-a^{\frac{1}{2}}/2x^{\frac{3}{2}}}=-\frac{2(x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

Hence, $\rho=\frac{2(x+a)^{\frac{3}{2}}}{a^{\frac{1}{2}}}$ *(in magnitude.)*

If P (x, y) be any point on the parabola, focus S, then $SP=x+a$. Thus, the above formula becomes

$$\rho=\frac{2(SP)^{\frac{3}{2}}}{a^{\frac{1}{2}}}.$$

Ex. 2. Find the radius of curvature at any point P(x, y) of the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \; i.\,e., \; (b^2x^2+a^2y^2=a^2b^2).$$

Differentiating,

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0; \qquad \therefore \frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

$$\text{and} \quad \frac{d^2y}{dx^2}=-\frac{a^2yb^2-b^2xa^2\frac{dy}{dx}}{a^4y^2}=-\frac{a^2b^2y+\frac{b^4x^2}{y}}{a^4y^2}$$

$$=-\frac{b^2(a^2y^2+b^2x^2)}{a^2y^3}=-\frac{b^4}{a^2y^3}$$

$$\therefore \quad \rho = \frac{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = -\frac{\left(a^4y^2+b^4x^2\right)^{\frac{3}{2}}}{a^4b^4} \qquad (6)$$

Note 1. y being considered positive, the negative sign for ρ is due to the sign of $\frac{d^2y}{dx^2}$. This implies that in the first and the second quadrants, the curve is convex upwards. The curve is concave upwards when $\frac{d^2y}{dx^2}$ is positive in the 3rd and the 4th quadrants where y is negative.

Note 2. $\frac{Xx}{a^2}+\frac{Yy}{b^2}=1$ is the tangent at P. The *perpendicular* p from the centre C upon the tangent is, therefore, given by

$$p^2 = \frac{1}{\frac{x^2}{a^4}+\frac{y^2}{b^4}} = \frac{a^4b^4}{b^4x^2+a^4y^2} \quad \therefore \quad \frac{a^6b^6}{p^3} = (b^4x^2+a^4y^2)^{\frac{3}{2}}$$

Hence from Ex. 1, $\rho = \frac{a^2b^2}{p^3}$ (*in magnitude*).

Ex. 3. Find ρ at any point of the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ (*four cusped hypocycloid*).

Differentiating, we have

$$\frac{2}{3}\,x^{-\frac{1}{3}}+\frac{2}{3}\,y^{-\frac{1}{3}}\frac{dy}{dx}=0$$

$$\therefore \frac{dy}{dx} = -\frac{x^{-\frac{1}{3}}}{y^{-\frac{1}{3}}} = -\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}}$$

$$\frac{d^2y}{dx^2} = -\frac{x^{\frac{1}{3}}\cdot\frac{1}{3}y^{-\frac{2}{3}}\frac{dy}{dx} - y^{\frac{1}{3}}\frac{1}{3}x^{-\frac{2}{3}}}{x^{\frac{2}{3}}}$$

$$= \frac{x^{\frac{1}{3}} \cdot y^{-\frac{2}{3}} \times \left(\frac{y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \right) + y^{\frac{1}{3}} x^{-\frac{2}{3}}}{3x^{\frac{2}{3}}}$$

$$= \frac{y^{\frac{1}{3}} \left(\frac{1}{y^{\frac{2}{3}}} + \frac{1}{x^{\frac{2}{3}}} \right)}{3x^{\frac{4}{3}} y^{\frac{2}{3}}} = \frac{a^{\frac{2}{3}}}{3x^{\frac{4}{3}} y^{\frac{1}{3}}}$$

$$\therefore \quad \rho = \frac{\left\{ 1 + \left(\frac{dy}{dx} \right)^2 \right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{\left\{ 1 + \left(\frac{-y^{\frac{1}{3}}}{x^{\frac{1}{3}}} \right)^2 \right\}^{\frac{3}{2}}}{a^{\frac{2}{3}} / 3x^{\frac{1}{3}} y^{\frac{1}{3}}}$$

$$= 3a^{\frac{1}{3}} x^{\frac{1}{3}} y^{\frac{1}{3}}$$

$$= 3(axy)^{\frac{1}{3}}.$$

Alternative method.

Let $x = a \cos^3 t$, $y = a \sin^3 t$, then

$$\frac{dx}{dt} = -3a \cos^2 t \sin t, \quad \frac{dy}{dt} = 3a \sin^2 t \cos t$$

$$\frac{d^2x}{dt^2} = 3a \cos t \, (2 \sin^2 t - \cos^2 t)$$

$$\frac{d^2y}{dt^2} = 3a \sin t (2 \cos^2 t - \sin^2 t)$$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dt} \Big/ \frac{dx}{dt} = -\tan t$$

$$\text{and } \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}}{\left(\frac{dx}{dt} \right)^3} = \frac{-9a^2 \sin^2 t \cos^2 t}{-27a^3 \cos^6 t \sin^3 t}$$

$$= + \frac{1}{3a \sin t \cos^4 t}$$

$$\therefore \quad \rho=\frac{\{1+(-\tan t)^2\}^{\frac{3}{2}}}{1/3a \sin t \cos^4 t}=+3a \sin t \cos t=3(axy)^{\frac{1}{3}}.$$

Note. If we determine ρ, by formula (5) p. 315, we get $\rho=-3a \sin t \cos t$ (*not* $+3a \sin t \cos t$).

EXAMPLES XLIII

1. Find ρ for the parabola $y^2=4ax$; $x=2at$, $y=at^2$.

2. Find ρ for the circle $x^2+y^2=2ax$.

3. Prove that for the ellipse $x=a \cos \phi$, $y=b \sin \phi$,

$$\rho=\frac{(a^2 \sin^2 \phi+b^2 \cos^2 \phi)^{\frac{3}{2}}}{ab}=\frac{(CD)^3}{ab},$$ where C is the centre of the ellipse and CD is the diameter conjugate to CP, P being the point $(a \cos \phi, b \sin \phi)$.

4. Prove that the radius of curvature of the catenary of uniform strength $y=c \log \frac{a}{c}$ is $c \sec \frac{x}{c}$.

5. Show that the radius of curvature of the catenary varies directly as the square of the ordinate, its equation being given in the form $y=c \cos h \frac{x}{c}$ or $y^2=c^2+s^2$.

6. Obtain ρ for the cycloid $x=a(\theta+\sin \theta)$, $y=a(1-\cos \theta)$; $s=4a \sin \psi$; $8ay=s^2$ and show that it is twice the intercept on the normal between the curve and the line $y=2a$.

7. Show that for the curve $y=\frac{ax}{a+x}$

$$\left(\frac{2\rho}{a}\right)^{\frac{2}{3}}=\left(\frac{y}{x}\right)^2+\left(\frac{x}{y}\right)^2.$$

8. Find the radius of the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1; \; x=a \cosh u, \; y=b \sinh u.$$

9. Find the co-ordinates of the points on the parabola $y^2=4ax$ for which radius of curvature is equal to the latus rectum.

10. Find ρ for the curve $\sqrt{x}+\sqrt{y}=1$ at $(\frac{1}{4}, \frac{1}{4})$.

11. Show that the circles of curvature of the parabola $y^2=4ax$ for the ends of the latus rectum have for their equations $x^2+y^2-10ax\pm4ay-3a^2=0$ and that they cut the curve again in the points $(9a, \mp6a)$.

12. Find ρ for each of the following curves :

(*i*) $x^4+y^4=2$ at (1, 1). (*ii*) $y=x+3x^2-x^3$ at (0, 0).

13. Find ρ for the curve $x=a\cos^3\phi$, $y=a\sin^3\phi$;

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

14. Find ρ for (*i*) $e^x=\sin y$. (*ii*) $y=e^x$ where it crosses the y-axis.

15. Show that for any curve

$$\rho^2=\frac{dx}{d\psi}\frac{d^2y}{d\psi^2}-\frac{dy}{d\psi}\frac{d^2x}{d\psi^2}.$$

16. Prove that if at every point of a curve

$$\frac{1}{\rho^4}=\left(\frac{d^3x}{ds^3}\right)^2+\left(\frac{d^3y}{ds^3}\right)^2$$ then the curve must be a circle.

17. If N denotes the length of the normal PG, show that

$$\frac{\rho}{N}=-\frac{1+y_1^2}{yy_2}.$$

4. (*a*) **Polar Form.** *To find the radius of curvature of the curve* $r=f(\theta)$ or $f(r, \theta)=0$.

$$\psi=\theta+\phi.$$

$$\therefore \quad \rho=\frac{d\psi}{ds}=\frac{d\psi}{d\theta}\cdot\frac{d\theta}{ds}=\left(1+\frac{d\phi}{d\theta}\right)\frac{d\theta}{ds}.$$

Now $\tan\phi=r\frac{d\theta}{dr}=\frac{r}{\frac{dr}{d\theta}}$; differentiating, we get

$$\sec^2\phi\frac{d\phi}{d\theta}=\frac{\left(\frac{dr}{d\theta}\right)-r\frac{d^2r}{d\theta^2}}{\left(\frac{dr}{d\theta}\right)^2}\quad\therefore\quad\frac{d\phi}{d\theta}=\frac{\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}{r^2+\left(\frac{dr}{d\theta}\right)^2}$$

$$\text{Also}\ \frac{ds}{d\theta}=\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}$$

$$\text{Hence}\ \rho=\frac{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2+2\left(\frac{dr}{d\theta}\right)^2-r\frac{d^2r}{d\theta^2}}\qquad(7)$$

Ex. Prove that for the curve $r^m=a^m\cos m\theta$

$$\rho=\frac{a^m}{(m+1)r^{m-1}}.$$

Examine the particular cases of the rectangular hyperbola lemniscate, parabola, cardioid ; straight line and circle.

Differentiating $r^m=a^m\cos m\theta$ logarithmically, we get

$$\frac{m}{r}\frac{dr}{d\theta}=-m\frac{\sin m\theta}{\cos m\theta};\ \text{i.e.,}\ \frac{dr}{d\theta}=-r\tan m\theta$$

$$\therefore\ \frac{d^2r}{d\theta^2}=-\frac{dr}{d\theta}\tan m\theta-mr\sec^2 m\theta=r\tan^2 m\theta-mr\sec^2 m\theta$$

$$\therefore\quad\rho=\frac{\{r^2+r^2\tan^2 m\theta\}^{\frac{3}{2}}}{r^2+2r^2\tan^2 m\theta-r^2\tan^2 m\theta+mr^2\sec^2 m\theta}$$

$$=\frac{r^3\sec^3 m\theta}{(m+1)\,r^2\sec^2 m\theta}=\frac{r}{(m+1)\cos m\theta}=\frac{r}{(m+1)\frac{r^m}{a^m}}$$

$$\text{Hence,}\ \rho=\frac{a^m}{(m+1)\,r^{m-1}}.$$

Putting in succession, $m=-2, 2, -\frac{1}{2}, \frac{1}{2}, -1, 1$ in the formula for ρ, we get the *radius of curvature*, respectively of the rectangular hyperbola, lemniscate, parabola, cardioid straight line and circle.

(*b*) **Pedal Form.** *To find the radius of curvature of the curve* $r=f(p)$ or $f(r, p)=0$; *i.e.*, to find ρ for a curve whose pedal equation is given.

$$p=r\sin\phi \text{ and } \psi=\theta+\phi$$

$$\therefore \frac{dp}{dr}=\sin\phi+r\cos\phi\frac{d\phi}{dr}=\frac{p}{r}+r\frac{d\phi}{ds}, \quad \because \cos\phi=\frac{dr}{ds}.$$

$$\text{And } \frac{d\psi}{ds}=\frac{d\theta}{ds}+\frac{d\phi}{ds}=\frac{p}{r^2}+\frac{1}{r}\frac{dp}{dr}-\frac{p}{r^2}=\frac{1}{r}\frac{dp}{dr}.$$

$$\therefore \quad \boldsymbol{\rho=r\frac{dr}{dp}}.$$

Note. If $u=\frac{1}{r}$, $\rho=-\frac{1}{u^2}\frac{du}{dp}$. ...(8)

5. Polar Tangential Form. *To find the radius of curvature of the curve* $p=f(\psi)$.

If w be the angle which the perpendicular from the pole on the tangent makes with the x-axis, $w=-\left(\frac{\pi}{2}-\psi\right)$.

The equation of the tangent is $x\cos w+y\sin w=p$, *i.e.*, $p=x\sin\psi-y\cos\psi$.

$$\frac{dp}{d\psi}=x\cos\psi+\sin\psi\frac{dx}{d\psi}+y\sin\psi-\cos\psi\frac{dy}{d\psi}$$

and $$\frac{dx}{d\psi}=\frac{dx}{ds}\cdot\frac{ds}{d\psi}=\rho\cos\psi$$

$$\frac{dy}{d\psi}=\frac{dy}{ds}\cdot\frac{ds}{d\psi}=\rho\sin\psi$$

$$\therefore \quad \frac{dp}{d\psi}=x\cos\psi+y\sin\psi$$

$$\therefore \quad \frac{d^2p}{d\psi^2}=\frac{dx}{d\psi}\cos\psi-x\sin\psi+\frac{dy}{d\psi}\sin\psi+y\cos\psi$$

$$=\rho(\cos^2\psi+\sin^2\psi)-p=\rho-p.$$

Hence $$\boldsymbol{\rho=p+\frac{d^2p}{d\psi^2}}.$$ ...(9)

EXAMPLES XLIV

1. Obtain ρ for the cardioid $r=a(1-\cos\theta)$, $2ap^2=r^3$; for the Lemniscate $r^2=a^2\sin 2\theta$ or $pa^2=r^3$.

2. Show that at any point on the equiangular spiral

$$r=ae^{\theta\cot\alpha}.$$

$\rho=r$ cosec α and that it subtends a right angle at the pole.

Find ρ for each of the following curves :

3. $r=a\cos m\theta$ at $\theta=0$.

4. $r^2=a^2\sin 2\theta$.

5. $pa^2=r^3$ [lemniscate].

6. $r^m=a^m\cos m\theta$.

7. $r^2+3p^2=k^2$.

8. $p^2=a^2\cos^2\psi-b^2\sin^2\psi$.

9. $r^2=\dfrac{a^2b^2}{p^2}+a^2-b^2$ (hyperbola).

10. Find ρ for the circle $r^2=2ap$ and show that its polar-tangential equation is $p=a(1+\sin\psi)$.

Obtain ρ from this equation also.

11. Find ρ for the parabola $p^2=ar$, $p\sin\psi=a$.

12. Find ρ for the ellipse (*i*) $a^2+b^2-r^2=\dfrac{a^2b^2}{p^2}$,

(*ii*) $\dfrac{b^2}{p^2}=\dfrac{2a}{r}-1$, (*iii*) $p^2=a^2\cos^2\psi+b^2\sin^2\psi$.

13. Show that the radius of curvature of the curve $r=a\sin n\theta$ at the origin is $\dfrac{na}{2}$.

14. Deduce the formula (7) from the formula (3) or (5).

[*Hint.* $x=r\cos\theta$ and $y=r\sin\theta$.]

15. Deduce the formula (7) from the formula (8).

[*Hint.* $\dfrac{1}{p^2}=u^2+\left(\dfrac{du}{d\theta}\right)^2$ where $u=\dfrac{1}{r}$. Differentiate w.r.t. u.]

16. Deduce the formula (8) from the formula (3).

[*Hint.* Tangent $Y-y=y'(X-x)$, $p=\dfrac{xy'-y}{\sqrt{1+y'^2}}$ and $r=\sqrt{x^2+y^2}$.]

17. Deduce the formula (3) from the formula (8) or (7).

18. Show that the radius of curvature of the curve $f(r, \theta)=0$ is $\dfrac{r \operatorname{cosec} \phi}{\left(1+\dfrac{d\phi}{d\theta}\right)}$. Apply this result to show that the radius of curvature of the circle $r=a \cos \theta$ is $\dfrac{a}{2}$.

19. Show that for the curve $r=f(\theta)$, the curvature is given by $\left(\dfrac{d^2u}{d\theta^2}+u\right) \sin^3 \phi$, where $u=\dfrac{1}{r}$.

Deduce that the curvature is given by $\dfrac{1}{r}\dfrac{dp}{dr}$.

20. Prove that for a plane curve, $\dfrac{d^2r}{ds^2}=\dfrac{\sin^2\phi}{r}-\dfrac{\sin \phi}{\rho}$.

Discuss the conditions for maximum or minimum value of the radius vector.

[*Hint.* At such a point, $\dfrac{dr}{ds}=0$. The point must lie at the feet of the normals.]

21. Prove that for the curve $r=a\ (1+\cos\ \theta)$, ρ^2/r is constant.

22. In the curve $p=\dfrac{r^{n+1}}{a^n}$, show that the radius of curvature varies inversely as the $(n-1)$th power of the radius vector.

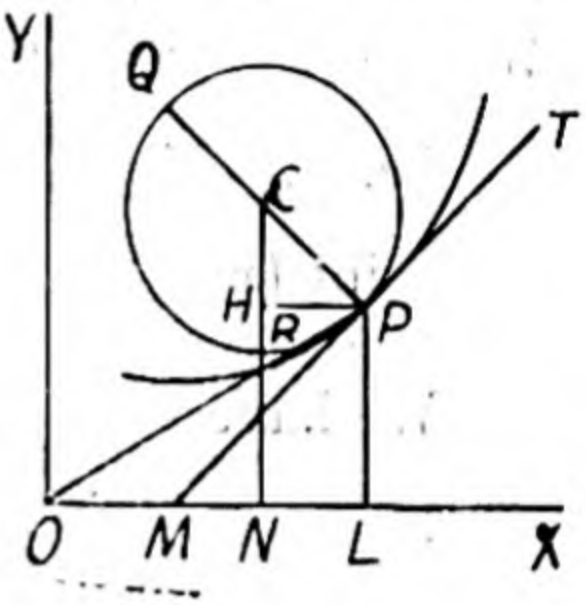
23. Show that in the Archimedian spiral $p^2=r^4/(r^2+a^2)$,

$$\rho=\frac{(r^2+a^2)^{\frac{3}{2}}}{r^2+2a^2}.$$

24. In the curve $\theta-\dfrac{\sqrt{r^2-a^2}}{a}=\cos^{-1}\dfrac{a}{r}$, prove that $\rho=\sqrt{r^2-a^2}$.

6. Co-ordinates of Centre of Curvature.

Let (x, y) be the co-ordinates of any point P on the curve and (ξ, η) those of the centre of curvature, C. Draw CN, PL $\perp$ OX and PH $\perp$ CN.



$\angle PCH = \psi$

$\xi = ON = OL - HP = x - \rho \sin \psi$

$\eta = NC = LP + HC = y + \rho \cos \psi$

(ψ being zero when the tangent is parallel to the x-axis).

Since $\tan \psi = \dfrac{dy}{dx}$,

$$\xi = x - \frac{y'(1+y'^2)}{y''}$$

$$\eta = y + \frac{1+y'^2}{y''}.$$

Ex. 1. Find the co-ordinates of the centre of curvature for any point (x, y) on the parabola $y^2 = 4ax$.

We have

$$y = 2a^{\frac{1}{2}} x^{\frac{1}{2}}, \quad \frac{dy}{dx} = a^{\frac{1}{2}} x^{-\frac{1}{2}}, \quad \frac{d^2y}{dx^2} = -\tfrac{1}{2} a^{\frac{1}{2}} x^{-\frac{3}{2}}$$

Hence, (1) $$\xi = x - \frac{\dfrac{dy}{dx}\left[1 + \left(\dfrac{dy}{dx}\right)^2\right]}{\dfrac{d^2y}{dx^2}} = x - \frac{a^{\frac{1}{2}} x^{-\frac{1}{2}} (1 + a x^{-1})}{-\tfrac{1}{2} a^{\frac{1}{2}} x^{-\frac{3}{2}}}$$

$$= x + 2(x+a) = 3x + 2a$$

(2) $$\eta = y + \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}} = y + \frac{1 + a x^{-1}}{-\tfrac{1}{2} a^{\frac{1}{2}} x^{-\frac{3}{2}}}$$

$$= 2a^{\frac{1}{2}} x^{\frac{1}{2}} - \frac{2(1 + a x - 1)}{a^{\frac{1}{2}} x^{-\frac{3}{2}}} = -2a^{-\frac{1}{2}} x^{\frac{3}{2}}$$

Corollary. Eliminating x between (1) and (2), we get $\frac{1}{27}(\xi-2a)^3=\frac{1}{4}(a^{\frac{1}{2}}\eta)^2$. The equation of the locus of the centre of curvature of the points on the parabola $y^2=4ax$ is $27ay^2=4(x-2a)^3$. (*Evolute of the parabola*).

Ex. 2. Find the co-ordinates of the centre of curvature at any point (x, y) of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

We have

$$(1)\quad b^2x^2+a^2y^2=a^2b^2,\quad \frac{dy}{dx}=\frac{-b^2x}{a^2y}$$

$$\frac{d^2y}{dx^2}=-\frac{b^2}{a^2}\frac{y-x\frac{dy}{dx}}{y^2}=\frac{-b^2}{a^2y^2}\left(y+\frac{b^2x^2}{a^2y}\right)=\frac{-b^4}{a^2y^3}.$$

$$(2)\quad \xi=x-\frac{y'(1+y'^2)}{y''}=x-\frac{\frac{-b^2x}{a^2y}\left(1+\frac{b^4x^2}{a^4y^2}\right)}{-b^4/y^3}$$

$$=x-\frac{x}{b^2}\left(\frac{a^4y^2+b^4x^2}{a^4}\right)$$

$$=x-\frac{x}{b^2}\frac{a^2(a^2b^2-b^2x^2)+b^4x^2}{a^4}=x-x\frac{a^2(a^2-x^2)+b^2x^2}{a^4}$$

$$=x-x+\frac{x^3(a^2-b^2)}{a^4}=\frac{a^2-b^2}{a^4}x^3.$$

$$(3)\quad \eta=y+\frac{1+y'^2}{y''}=y-\frac{(a^4y^2+b^4x^2)y}{a^2b^4}$$

$$=y-\left(\frac{a^2y^2}{b^4}+\frac{x^2}{a^2}\right)y=y-\left\{\frac{a^2y^2}{b^4}+\left(1-\frac{y^2}{b^2}\right)\right\}y.$$

$$=-\frac{a^2-b^3}{b^4}y^3.$$

Corollary. Eliminating x and y between (1), (2) and (3), we obtain the locus described by (ξ, η) in the form $(a\xi)^{\frac{2}{3}}+(b\eta)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$. (*Evolute of the ellipse*).

Alternative solution.

We have, $x=a\cos\phi$, $y=b\sin\phi$;

$$\therefore\ \frac{dx}{d\phi}=-a\sin\phi,\ \frac{dy}{d\phi}=b\cos\phi,\ \frac{d^2x}{d\phi^2}=-a\cos\phi,\ \frac{d^2y}{d\phi^2}=-b\sin\phi.$$

Hence, $\dfrac{dy}{dx}=-\dfrac{b}{a}\cot\phi$

$$\frac{d^2y}{dx^2}=\frac{\left(\frac{dx}{d\phi}\frac{d^2y}{d\phi}-\frac{dy}{d\phi}\frac{d^2x}{d\phi^2}\right)}{\left(\frac{dx}{d\phi}\right)^3}=\frac{ab}{-a^3\sin^3\phi}.$$

Hence $\xi=x-\dfrac{y'(1+y'^2)}{y''}=x-\dfrac{-\frac{b}{a}\cot\phi\left(1+\frac{b^2}{a^2}\cot^2\phi\right)}{\frac{ab}{-a^3\sin^3\phi}}$

$$=a\cos\phi-\frac{b\cos\phi\,(a^2\sin^2\phi+b^2\cos^2\phi)}{ab}$$

i.e., $$\xi=\frac{a^2-b^2}{a}\cos^3\phi=\frac{a^2-b^2}{a^4}x^3 \qquad \ldots\ldots(1)$$

$$\eta=y+\frac{1+y'^2}{y''}=y+\frac{1+\frac{b^2}{a^2}\cot^2\phi}{\frac{ab}{-a^3\sin^3\phi}}$$

$$=b\sin\phi+\frac{(a^2\sin^2\phi+b^2\cos^2\phi)\times(-\sin\phi)}{b}$$

$$=\frac{b^2-a^2\sin^2\phi-b^2\cos^2\phi}{b}\times\sin\phi$$

or $$\eta=\frac{b^2-a^2}{b}\sin^3\phi=\frac{b^2-a^2}{b^4}y^3 \qquad \ldots\ldots(2)$$

From (1), we have

$\left(\dfrac{a\xi}{a^2-b^2}\right)^{\frac{2}{3}}=\dfrac{x^2}{a^2}$; similarly from (2), $\left(\dfrac{b\eta}{a^2-b^2}\right)^{\frac{2}{3}}$

$$=\left\{\left(-\frac{y^3}{b^3}\right)^{\frac{1}{3}}\right\}^2=\frac{y^2}{b^2}.$$

$$\therefore \quad \left(\frac{a\xi}{a^2-b^2}\right)^{\frac{2}{3}}+\left(\frac{b\eta}{a^2-b^2}\right)^{\frac{2}{3}}=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Hence the *locus of the centre of curvature of the ellipse* (*called Evolute of the ellipse*) is $(a\xi)^{\frac{2}{3}}+(b\eta]^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

EXAMPLES XLIV (A)

Show that co-ordinates of the centre of curvature of the following curves are as given below :

1. The semicubical parabola $x^3=ay^2$

$$\xi=-x-\frac{9x^2}{2a}, \quad \eta=4\left(x+\frac{a}{3}\right)\sqrt{\frac{x}{a}}.$$

2. The four cusped hypocycloid $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$

$$\xi=x+3(xy^2)^{\frac{1}{3}}=x+3x^{\frac{1}{3}}y^{\frac{2}{3}}.$$
$$\eta=y+3(x^2y)^{\frac{1}{3}}=y+3x^{\frac{2}{3}}y^{\frac{1}{3}}.$$

3. The catenary $y=\frac{c}{2}\left(e^{\frac{x}{c}}+e^{-\frac{x}{c}}\right)$

$$\xi=x-\frac{y}{c}\sqrt{y^2-c^2}, \quad \eta=2y.$$

4. The equilateral hyperbola $xy=a^2$

$$2\xi=3x+\frac{y^2}{x}, \quad 2\eta=3y+\frac{x^2}{y}$$

and deduce that

$$\xi+\eta=\frac{a}{2}\left(\frac{x}{a}+\frac{a}{x}\right)^2, \quad \xi-\eta=\frac{a}{2}\left(\frac{a}{x}-\frac{x}{a}\right)^2.$$

5. Show that in the parabola

$$\sqrt{x}+\sqrt{y}=\sqrt{a}, \quad \xi+\eta=3(x+y).$$

(*b*) Length of Chord of Curvature.

Any chord through P of the circle of curvature at P is called a chord of curvature of the curve.

1. If the chord of curvature PR passes through the origin, $PR=2\rho \sin\phi$; $[\phi=\angle MPO]$.

$$=2\rho\frac{p}{r}=2p\frac{dr}{dp}.$$

2. If the equation of the curve be $p=f(r)$,

$$\text{PR}=2f(r)/f'(r).$$

3. If the equation of the curve be $r=f(\theta)$,

$$\sin\phi=r\frac{d\theta}{ds} \text{ and } \frac{ds}{d\theta}=\sqrt{\left(\frac{dr}{d\theta}\right)^2+r^2}$$

$$\therefore \text{PR}=2\rho\ r\frac{d\theta}{ds}=\frac{2r(r^2+r'^2)}{r^2+2r'^2-rr''}.$$

4. If the equation of the curve be $y=f(x)$,

$$\psi=\theta+\phi \text{ and } \tan\psi=y'$$

$$\therefore \text{PR}=2\rho\sin(\psi-\theta]=\frac{2\rho}{r}(x\sin\psi-y\cos\psi)$$

$$=\frac{2(x-y')(1+y'^2)}{y''\sqrt{x^2+y^2}}.$$

If the chord does not pass through the origin,

$$\text{PR}=2\rho\sin\alpha,$$

where α is the angle made by the chord with the tangent at P.

Ex. 1. Prove that the points on the curve $r=f(\theta)$, the circles of curvature at which pass through the origin are given by the equation

$$f(\theta)+f''(\theta)=0.$$

Let P be such a point. OP$=r=$chord of curvature.

$\therefore\ r=2r(r^2+r'^2)/(r^2+2r'^2-rr'')$

i.e., $r^2+rr''=0$. *i.e.*, $r+r''=0$,

whence $f(\theta)+f''(\theta)=0$.

7. *To show that the centre of curvature is the limiting position of the point of intersection of two consecutive normals.*

Let P(x, y) and Q$(x+\delta x,\ y+\delta y)$ be two consecutive points on the curve $y=\phi(x)$. Then

(1) $(X-x)+(Y-y)\frac{dy}{dx}=0$ is the normal at P(x, y).

(2) $(X-x-\delta x)+(Y-y-\delta y)\left(\frac{dy}{dx}\right)_{x+\delta x,\ y+\delta y}=0$

is the normal at Q$(x+\delta x, y+\delta y)$.

Put $\frac{dy}{dx}=f(x)$; then $\left(\frac{dy}{dx}\right)_{x+\delta x,\ y+\delta y}=f(x+\delta x)$.

By subtraction, the *point of intersection* of the normals (1) and (2) is given by

$$\delta x+\delta y f(x+\delta x)=(\mathrm{Y}-y)\,[f\,(x+\delta x)-f\,(x)]$$

$$\therefore\quad \mathrm{Y}-y=\frac{\delta x+\delta y\, f(x+\delta x)}{f(x+\delta x]-f\,(x)}=\frac{1+\dfrac{\delta y}{\delta x}f\,(x+\delta x)}{\dfrac{f(x+\delta x)-f(x)}{\delta x}}.$$

Proceeding to the limit, we get

$$\left.\begin{aligned}\eta-y&=\frac{1+\dfrac{dy}{dx}f(x)}{f'\,(x)}\\ \text{and.}\ \therefore\xi-x&=-(\eta-y)f(x)\end{aligned}\right\}$$

where $f(x)=\frac{dy}{dx}$ and $f'\,(x)=\frac{d^2y}{dx^2}$ and ξ, η is the limiting position of X and Y the point of intersection of the normals.

$$\eta=y+\frac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}};\qquad \xi=x-\frac{1+\left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}\times\frac{dy}{dx}.$$

The expressions on the right are the co-ordinates of the centre of curvature and those on the left are *co-ordinates of the limiting position of the point of intersection of the normals.*

***8.** A curve whose equation can be put in the form $x=f(t)$ $y=\phi(t)$ is touched at a point P and cut at a neighbouring point Q by a circle with centre O. Show that if C, the centre of curvature at P, exists, then O$\rightarrow$C as Q$\rightarrow$P.

Since C exists, $f'(t)$ and $\phi'(t)$ exist and are not both zero.

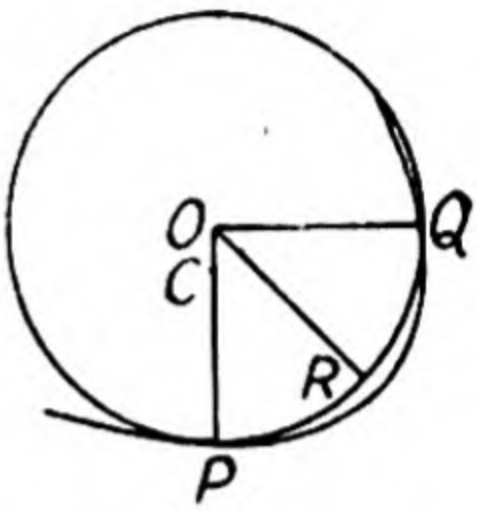
If O be (α, β) and R any point on the infinitesimal are PQ.

$$\mathrm{OR}^2=[f(t)-\alpha]^2+[\phi(t)-\beta]^2=\mathrm{F}(t).$$

$\therefore$ F(t) is continuous and possesses a derivative at every point of the are and since F(t) at P=F(t) at Q, by Rolle's

Theorem $F'(t)=0$ at some point R of the arc between P and Q.

Hence $f'(t)\ [f(t)-\alpha]+\phi'(t)[\phi(t)-\beta]=0$.



And as $f'(t)$ and $\phi'(t)$ are not both zero at any point of the arc PQ, OR is therefore normal to the curve at R.

Now R→P as Q→P and by Art. 7. O→C as R→P.

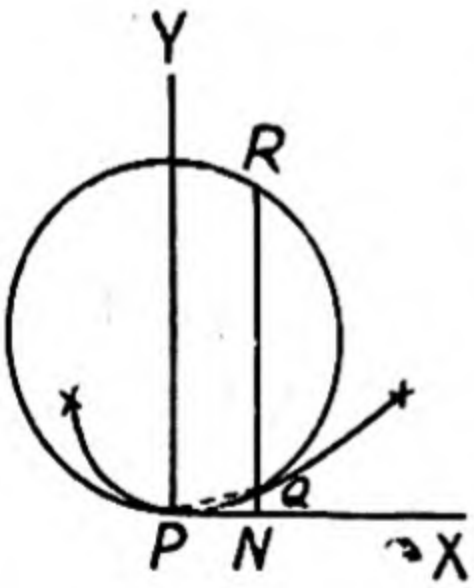
9. Curvature at Origin. *Newton's Method.*

(*a*) This method leads to a single formula for the radius of curvature and *needs no differentiation.*

In this method the axes chosen are the tangent and the normal at the point at which the curvature is required.

Taking the tangent and the normal at P as axes and assuming the existence of ρ at P, we will prove that

$$\rho=\underset{x\to 0}{\text{Lt}}\ \frac{x^2}{2y} \text{ or } \rho=\underset{y\to 0}{\text{Lt}}\ \frac{y^2}{2x}.$$



Proof (*i*) If the tangent and the normal at P be the axes of x and y and Q be (x, y), draw a circle to touch the curve at P and to pass through Q.

By Art. 7, this circle tends to the circle of curvature at P as Q →P.

Now $NP^2=NQ.NR$, *i.e.*, $r^2=y(2r-y)$ where r is the radius of this circle.

$$\therefore\quad \frac{x^2}{2y}=r-\frac{1}{2} \text{ which tends to}$$

ρ when $x\to 0$.

Hence $\rho=\underset{x\to 0}{\text{Lt}}\ \frac{x^2}{2y}$.

(*ii*) Similarly if the tangent and the normal at P be the axes of y and x, then $\rho = \underset{y\to 0}{\text{Lt}} \frac{y^2}{2x}$.

In polar co-ordinates $\rho = \underset{x\to 0}{\text{Lt}} \frac{x^2}{2y} = \text{Lt} \frac{r^2\cos^2\theta}{2r\sin\theta}$

$= \underset{\theta\to 0}{\text{Lt}} \left(\frac{r}{2\theta}\right)$.

(*b*) The same method may be applied when the tangent to the curve does not coincide with one of the axes. We have to calculate the length of the perpendicular QN from a point Q(x, y) on the curve in the vicinity of the origin to the tangent at the origin and the distance ON of the foot N of this perpendicular from the origin and then to find the limit of $\frac{ON^2}{2QN}$. If this limit exists, then it is equal to ρ at O. ρ is ultimately Lt $\frac{x^2+y^2}{2(x\sin\alpha - y\cos\alpha)}$ where $y = x\tan\alpha$ is the tangent at the origin.

10. The ordinary Cartesian formula gives the radius of curvature at the origin, for we have to calculate $\left(\frac{dy}{dx}\right)_{x=0}$. $\left(\frac{d^2y}{dx^2}\right)_{x=0}$ but the following method is easy and simple in comparison with this method as well as with Newton's method when the tangent at the origin is not an axis.

Let it be possible to expand y easily in powers of x, so that $y = px + q\frac{x^2}{\underline{|2}} + \ldots$., where we know from Maclaurin's Theorem that

$$p = \left(\frac{dy}{dx}\right)_{x=0,\ y=0} \text{ and } q = \left(\frac{d^2y}{dx^2}\right)_{x=0,\ y=0}.$$

Substitute for y the expression $px + q\frac{x^2}{\underline{|2}} + \ldots\ldots\ldots$ and equate co-efficients of like powers in the identity so obtained. This gives p and q.

The radius of curvature at the origin will then be

$$\rho=\pm\frac{(1+p^2)^{\frac{3}{2}}}{q}, \text{ at any point.}$$

To find the curvature at any point by this method, shift the origin to that point.

This method will also give *the radii of curvature at a double point* when it is taken as the origin.

Ex. 1. Show that the radii of curvature at the origin for the curve $x^3+y^3=3axy$ are each equal to $\frac{3a}{2}$.

The origin is a real node, the axes being the tangents thereat.

$$\therefore \quad \rho_1=\text{Lt}\frac{x^2}{2y} \text{ and } \rho_2=\text{Lt}\frac{y^2}{2x}.$$

To find ρ_1, replace y by $\frac{x^2}{2\rho_1}$ in the equation. Then

$$x^3+\frac{x^6}{8{\rho_1}^3}=3ax\frac{x^2}{2\rho_1} \text{ approximately;}$$

$$\therefore \quad 1+\frac{x^3}{8{\rho_1}^3}=\frac{3a}{2\rho_1}. \quad \text{When } x\to 0,\ \rho_1=\frac{3a}{2}$$

To find ρ_2, replace x by $\frac{y^2}{2\rho_2}$ in the equation. Then

$$\frac{y^6}{8{\rho_2}^3}+y^3=3a\frac{y^2}{2\rho_2}.\ y \text{ approximately.}$$

$$\therefore \quad \frac{y^3}{8{\rho_2}^3}+1=\frac{3a}{2\rho_2}. \quad \text{When } x\to 0,\ \rho_2=\frac{3a}{2}.$$

Note. The usual formula (4) cannot be used in this case as both f_x and f_y are zero at the origin which is a double point and ρ becomes indeterminate.

Ex. 2. Find the radii of curvature of the curve $y=x^2(x-3)$ at the points where the tangent is parallel to the x-axis.

Since the tangents have to be parallel to the x-axis $\frac{dy}{dx}=0$, *i. e.*, $3x^2-6x=0$ which gives $x=0$, $x=2$.

$\therefore$ the tangents at (0, 0) and (2, -4) are parallel to the x-axis.

The x-axis is a tangent to the curve at the origin ;

$\therefore$ the radius of curvature at the origin

$$=\frac{1}{2}\operatorname*{Lt}_{x\to 0}\frac{x^2}{y}=\frac{1}{2}\operatorname*{Lt}\frac{1}{x-3}=-\frac{1}{6}.$$

In order to find the radius of curvature at (2, -4) shift the origin to the point and the equation of the curve takes the form :

$$y-4=(x+2)^2(x-1) \text{ or } y=x^2(x+3).$$

The tangent at the origin is now the new x-axis

$\therefore$ the radius of curvature there

$$=\frac{1}{2}\operatorname*{Lt}\frac{x^2}{y}=\frac{1}{2}\operatorname*{Lt}\frac{1}{x+3}=\frac{1}{6}.$$

Ex. 3. Find the radii of curvature at the origin of the curve

$$x^5+ax^2y^2-ax^3y-2a^2xy^2+a^2y^3=0.$$

The tangents at the origin which is double point are given by the equation

$$2xy^2-y^3=0.$$

$y=0$ is a cuspidal tangent, $y=2x$ is a tangent to the 3rd branch of the curve.

Put $y=px+\frac{qx^3}{\lfloor 2}+..$ in the equation of the curve.

Then

$$x^5+ax^2\left(px+q\frac{x^3}{\lfloor 2}+..\right)^2-ax^2\left(px+q^3\frac{x^2}{\lfloor 2}+..\right)$$

$$-a^2x\left(px+q\frac{x^2}{\lfloor 2}+...\right)^2+a^2\left(px+q\frac{x^2}{\lfloor 2}+...\right)=0.$$

Equating the co-efficients of x^3, x^4 on both sides, we get

$$-2a^2p^2+a^2p^3=0. \qquad p^2=0 \text{ or } p=2,$$

and $ap^2-ap-2a^2pq+\frac{3}{2}a^2p^2q=0.$

(*i*) When $p=0$, this relation does not give q.

(*ii*) But when $p=2$, $4-2-2a.\ 2q+\frac{3}{2}a.\ 4q=0,$

i.e., $2aq+2=0$, *i.e.*, $q=-\frac{1}{a}$.

The radius of curvature corresponding to the values $p=2$, $q=-\frac{1}{a}$ is equal to $\frac{(1+q^2)^{\frac{3}{2}}}{q}=-5^{\frac{3}{2}}a.$

Note 1. In order to find the radii of curvature corresponding to $p^2=0$, put the co-officients of $p^5=0$. We get

$$1+apq-\frac{aq}{2}-\frac{a^2}{2}q^2+3\frac{a^2}{2}pq^2=0$$

$$i.e.,-\frac{a^2}{2}\ q^2-\frac{aq}{2}+1=0;$$

$$\therefore\ q=\frac{\frac{a}{2}\pm\sqrt{\left(\frac{a^4}{4}+2a^2\right)}}{-a^2}=-\frac{1}{2a}\mp\frac{2}{2a}=-\frac{2}{a},\ \frac{1}{a}.$$

The radii of curvature are given by $\frac{(1+p^2)^{\frac{3}{2}}}{q}$ and hence they are $a, -\frac{a}{2}$.

Note 2. Newton's method gives the radii of curvature corresponding to the tangent $y=0$ if we replace y by $\frac{x^2}{2\rho}$.

Thus we get

$$x^5+a\frac{a^6}{4\rho^2}-a\ \frac{x^5}{\rho 2}-a^2\frac{x^5}{2\rho^2}+a^3\frac{x^6}{8\rho^3}=0.$$

$$i.e.,\ 1+\frac{a\rho}{4\rho^2}-\frac{a}{2\rho}-\frac{a^2}{2\rho^2}+\frac{a^2x}{8\rho^3}=0.$$

Let $x\to 0$; $\frac{a^2}{2\rho^3}+\frac{a}{2\rho}-1=0 \quad \therefore \rho=-\frac{a}{2}, a,$

showing that one branch touching the x-axis lies above the x-axis and the other below it.

EXAMPLES XLV

Find by Newton's method the radius of curvature at the origin for each of the following curves:

1. $2x^2+y^2=2y$. **2.** $y^2(a^2-x^2)=a^3x$.

3. $x=a(\theta+\sin\theta)$, $y=a(1-\cos\theta)$.

4. Find the radii of curvature at the origin of the two branches of the curve given by the equations $x=1-t^2$, $y=t-t^3$.

5. $2x^4-5ax^2y-2axy^2+2a^2y^2=0$.

6. $ax^3-2b^2xy+cy^3=x^4+y^4$. **7.** $x^4+y^4=2a(x+y)$.

8. $2x^2+y^2=2(x-y)$. **9.** $r=a\sin n\theta$.

Find the radius of curvature at the origin of the curves.

10. $a(x^2-y^2)=2x^3+y^3$. **11.** $y^2-3xy-4x^2+x^3+xy+y^5=0$.

12. Show that the radius of curvature of the general curve $ax+by+(a_1x^2+2b_1xy+b_1y^2)+(a_2h^3..)+=0$ is

$$\tfrac{1}{2}\frac{(a^2+b^2)^{\frac{3}{2}}}{a_1b^2-2b_1ab+b_1a^2}.$$

[*Hint.* Put $y=px+\frac{qx^2}{2}+\ldots\ldots$].

13. Show that the chord of curvature through the pole for the curve $p=f(r)$ is $2f(r)/f'(r)$ or $2p\,dr/dp$.

14. Prove that the chord of curvature through the origin of the equiangular spiral $r=ae^{\theta\cot a}$ is equal to $2r$.

15. Show that the chord of curvature of the curve $r^m=a^m \cos m\theta$ through the pole is $\frac{2r}{m+1}$.

Examine the cases when $m=-1, 1$.

16. Show that the chord of curvature through the pole of the cardioid $r=a(1-\cos\theta)$, $r^3=2ap^2$ is $\frac{4}{3}r$.

17. Find the chord of curvature through the pole of the equiangular spiral $r=ae^{m\theta}$.

18. If C_x, C_y be the chords of curvature parallel to the axes at any point of the curve $y=ae^{\frac{x}{a}}$, prove that

$$\frac{1}{C_x^2}+\frac{1}{C_y^2}=\frac{1}{2aC_x}.$$

19. If the angle between the radius vector and the normal to a curve has a maximum or a minimum value, prove that the chord of curvature through the pole is $2r$.

20. Show that in the curve $y-b=c \log \sec \frac{x-a}{c}$, the chord of curvature parallel to the axis of y is $2c$ and find the radius of curvature at the point (a, b).

21. If the conics $3x^2+8y^2=24$, $x^2-4y=4$ have P as one of their points of intersection, find the radius of curvature of each of the curves at P, and show that the distance between the two centres of curvature is the square root of the sum of the squares of the radii of curvature.

22. P(r, p) and Q$(r+\delta r, p+\delta p)$ are two neighbouring points on a curve and a circle with centre R and radius k touches the curve at P and passes through Q. Show that $OR^2=k^2+r^2-2kp=k^2+(r+\delta r)^2-2k(p+\delta p)$ where O is the pole.

Deduce that the radius of curvature at $P=r\frac{dr}{dp}$.

23. Prove that the radius of curvature of the curve $a^{m-1}y=x^m$ at the origion is 0, $\frac{a}{2}$, ∞ according to circumstances.

24. Find the radius of curvature at the origin in the curve

$$3y=4x-15x^2-3x^3.$$

Find also at what points the radius of curvature is infinite.

25. Find the radii of curvature at the points at which tangents are parallel to the x-axis in each of the curves:

(*i*) $y=2x+3x^2-2xy+y^2$ (*ii*) $y=x-3x^2+3x^3$.

26. Find the radius of curvature of the curve:

$$x^4-3x^2y+7xy^2-2x^2-3y^2+6x=0,$$

at the points at which the tangents are parallel to either axis.

27. Find the radius of curvature at the origin in each of the curves (*i*) $y^3=x(x+y)$ (*ii*) $y^3=(x+y)^2$.

28. Show by Newton's method that the radius of curvature of the ellipse at P is $\frac{\beta^2}{p}$ where p is the perpendicular from the centre C to the tangent at P and β is the semi-diameter conjugate to CP.

29. Find the curvature at $(0, a)$ of the curve $(x^2+y^2)^2=a^2(y^2-x^2)$.

30. Show that the radii of curvature of the curve $y^2(a-x)=x^2(a+x)$ for the origin are $\pm a\sqrt{2}$ and for the point $(-a, 0)$ is $\frac{a}{4}$.

31. Find the points on the parabola $y^2=8x$ at which the radius of curvature is $7\frac{13}{16}$.

32. Regarding the centre of curvature of a curve as the ultimate intersection of the normals at two neighbouring points find the co-ordinates of the centre of curvature of the curves (*i*) $x=at^2$, $y=2at$ (*ii*) $x=a\cos\phi$, $y=b\sin\phi$.

Show that the circle of curvature in the latter case cuts the curve again at the point $(a\cos 3\phi, -b\sin 3\phi)$.

33. Show that the circle of curvature at the origin for the curve $x+y=ax^2+by^2+cx^3$ is $(a+b)(x^2+y^2)=2x+2y$.

34. Show that the circle of curvature at the point $(at^2, 2at)$ of the parabola $y^2=4ax$ has for its equation

$$x^2+y^2-6at^2x-4ax+4at^3-3a^2t^4=0$$

and find the point where it meets the curve again.

35. Show that the centre of curvature (α, β) of the curve $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is given by $\alpha = x + 3x^{\frac{1}{3}} y^{\frac{2}{3}}$, $\beta = y + 3x^{\frac{2}{3}} y^{\frac{1}{3}}$.

36. Prove that the centres of curvature at points of cycloid $x = a(t + \sin t)$, $y = a(1 + \cos t)$ lie on an equal cycloid.

37. Find the radii of curvature at the intersections of the hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $xy = c^2$ and deduce the condition that the curves may intersect each other orthogonally. What further condition must be satsified if these radii are equal in length?

38. A curve touches the x-axis at the origin O and P is the point (x, y). If the length of the chord OP is c, prove that $\underset{x \to 0}{\text{Lt}} \frac{c - x}{x^3} = \frac{1}{8\rho^2}$ where ρ is the radius of curvature at O.

39. Prove that for the curve $s = m(\sec^3\phi - 1)$, $\rho = 3m \tan\phi \sec^3\phi$ and hence that $3m \frac{dy}{dx} \frac{d^2y}{dx^2} = 1$. Also that this differential equation is satisfied by the curve $27 my^2 = 8x^3$.

40. Prove that for the curve $s = a \log \cot \left[\frac{\pi}{4} - \frac{\phi}{2} \right] + a \frac{\sin\phi}{\cos^2\phi}$, $\rho = 2a \sec^3\phi$ and hence that $\frac{d^2y}{dx^2} = \frac{1}{2a}$, and that this differential equation is satisfied by the curve $x^2 = 4ay$.

41. Show that the equation of a normal to the curve $y = f(x)$ can be put in the form $\frac{X - x}{\cos\theta} = \frac{Y - y}{\sin\theta} = r$, where $\tan\theta = -\cot\phi$. Hence deduce that (X, Y) the co-ordinates of the centre of curvature are given by

$$X = x + y_1 \frac{(1 - y_1^2)}{y_2} \text{ and } Y = y + \frac{1 + y_1^2}{y_2}.$$

42. Show that the co-ordinates of the centre of curvature of any curve are $\frac{1}{2} \frac{d^2r^2}{dy^2} \Big| \frac{d^2x}{dy^2}$, $\frac{1}{2} \frac{d^3r^2}{dx^2} \Big| \frac{d^2y}{dx^2}$.

Evolutes and Involutes

11. *The evolute of a given curve may be regarded as the locus of its centres of curvature or as the envelope of its normals.*

Any curve which has a given curve for evolute is called an involute of the given curve.

Thus a given curve is an involute of its evolute.

(*a*) *Assuming the evolute of a given curve* $y=\phi(x)$ *to be the ocus of its centres of curvature, show that it is the envelope of its normals.*

If (ξ, η) be the centre of curvature at $[\alpha, \phi(\alpha)]$ of the given curve, then

$$\xi=\alpha-\frac{\phi'(\alpha)\{1-[\phi'(\alpha)]^2\}}{\phi''(\alpha)}\ ;\ \eta=\phi(\alpha)+\frac{1+[\phi'(\alpha)]^2}{\phi''(\alpha)} \quad ...(1)$$

This gives α the locus of (ξ, η) which is obtained by eliminating α.

The equation of the normal at α is

$$x-\alpha+\{y-\phi(\alpha)\}\phi'(\alpha)=0 \quad ...(2)$$

The envelope of (2) is found by the usual method, α being a variable parameter.

$$-1-[\phi'(\alpha)]^2+\{y-\phi(\alpha)\}\phi''(\alpha)=0 \quad ...(3)$$

$$\therefore\ y=\phi(a)+\frac{1+[\phi'(\alpha)]^2}{\phi''(\alpha)} \text{ and } x=\alpha-\frac{\phi'(a)\{1+[\phi'(\alpha)]^2\}}{\phi''(\alpha)} \quad ...(4)$$

Elimination of α from (4) is the same as the elimination of α from (1). Hence the result.

Otherwise :—If a normal to a given curve touches a curve at C, then C is the limit of the point of intersection of neighbouring normals and therefore a centre of curvature of the given curve. All points on σ are therefore centres of curvature of the given curve.

(*b*) *Regarding the evolute of a given vurve* $y=\phi(x)$ *as the envelope of its normals, show that it is the locus of its centres of curvature.*

The normal at (x, y) of the given curve is

$$X-x+(Y-y)y'=0$$

i.e., $$X-x+\{Y-\phi(x)\}\phi'(x)=0 \quad ...(5)$$

Let the envelope of (5) be given.

If (ξ, η) be the centre of curvature at (x, y) of the curve $y=\phi(x)$, $\xi=x-\rho \sin\psi$, $\eta=y+\rho\cos\psi$

$$\therefore \quad \frac{d\xi}{dx}=1-\frac{d\rho}{dx}\sin\psi-\rho\cos\psi\frac{d\psi}{dx}$$

$$=1-\frac{d\rho}{dx}\cdot\frac{dy}{ds}-\frac{ds}{d\psi}\cdot\frac{dx}{ds}\cdot\frac{d\psi}{dx}=-\frac{d\rho}{dx}\cdot\frac{dy}{ds}.$$

Similarly $\frac{d\eta}{dx}=\frac{d\rho}{dx}\cdot\frac{dx}{ds}$.

$$\therefore \quad \frac{d\eta}{d\xi}=-\frac{dx}{dy}=-\cot\psi. \qquad \ldots(6)$$

$\therefore$ the equation to the tangent of the centre locus (ξ, η) at (ξ, η) is $Y-\eta=\frac{d\eta}{d\xi}(X-\xi)$

i.e., $(X-\xi)+(Y-\eta)\tan\psi=0$.

Evidently this line passes through (x, y).

$\therefore$ the equation to the tangent of the centre locus (ξ, η) at (ξ, η) may be written in the form :—

$$X-x+\{Y-\phi(x)\}\phi'(x)=0 \qquad \ldots(7)$$

(5) and (7) are identical.

Hence the tangent at every point of the centre-locus coincides with the normal at the corresponding point of the given curve. But a curve is the envelope of its tangents.

Thus the centre-locus coincides with the envelope of the normals.

We have proved the following :—

(*i*) The normal at any point of a given curve is a tangent to the evolute at the corresponding centre of the curvature ; and any tangent to the evolute is a normal to the curve.

(*ii*) The evolute touches each normal at the centre of curvature. See (6).

(*iii*) A curve which at every point touches a normal to a given curve is the evolute of the given curve, (the evolute being regarded as the locus of its centre of curvature).

12. *To show that the arc of the evolute, corresponding to an arc of a given curve for which* ρ *constantly increases or decreases, is equal to the difference of the radii of curvature touching its extremities.*

Let (x, y) be any point on the arc s of a given curve and (ξ, η) the corresponding centre of curvature and σ the arc of the evolute corresponding to s. Then

$$\xi = x - \rho \sin \psi, \quad \eta = y + \rho \cos \psi$$

$$\therefore \quad \frac{d\xi}{dx} = -\frac{dy}{ds}\frac{d\rho}{dx} \cdot \frac{d\eta}{dx} = \frac{d\rho}{dx} \cdot \frac{dx}{ds}$$

Squaring and adding ; $\left(\frac{d\sigma}{dx}\right)^2 = \left(\frac{d\rho}{dx}\right)^2$

$\therefore d\sigma = d\rho$ as $d\rho$ does not change sign, we can choose a suitable direction in which to measure σ or the positive or negative sign being taken as σ increases or decreases when ρ increases.

Hence $\sigma = \rho + c$ where c is a constant.

When $\sigma = 0$, $\rho = \rho_0$ and $\therefore \; c = -\rho_0$.

Hence $\sigma = \rho - \rho_0$.

Cor. Every plane curve whose curvature is constant is a circle.

For $d\sigma = 0$, $\therefore$ the centre of curvature at one point is the centre of curvature at any point of the curve. Hence every point of the curve lies on a fixed circle.

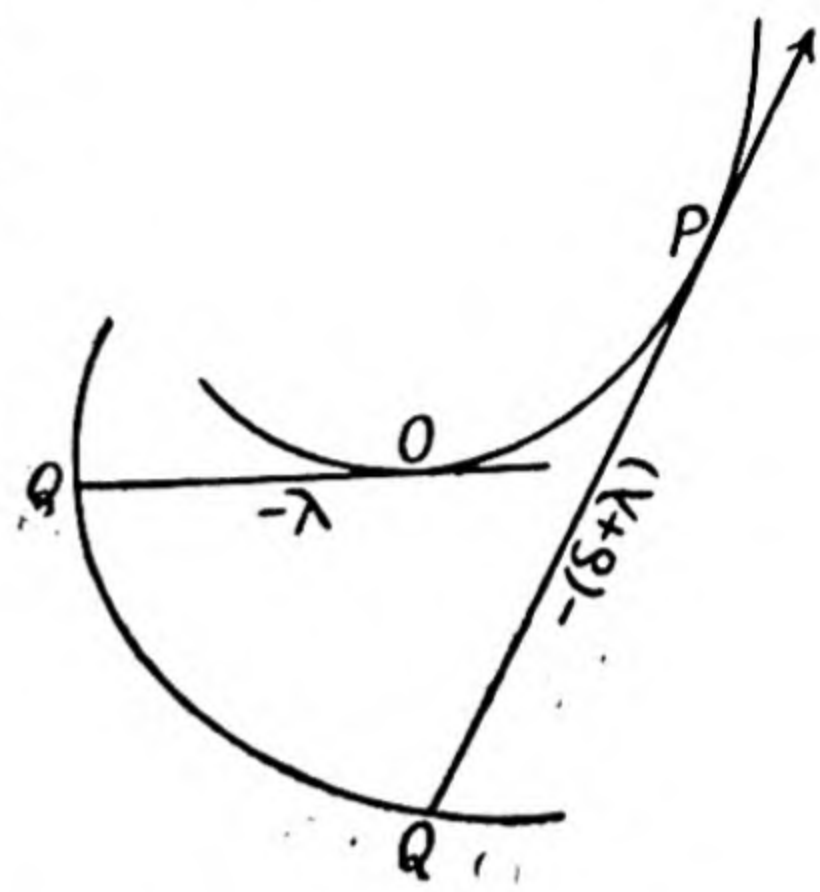
Cor. The radius of curvature of the evolute is equal to $\frac{d\rho}{d\psi}$.

If ρ' be the radius of curvature of the evolute and ψ' the angle made by the normal with the y-axis,

$$\rho' = \frac{d\sigma}{d\psi'} \text{ and } \psi' = \psi \; \therefore \; \rho' = \frac{d\rho}{d\psi} = \frac{d^2s}{d\psi^2} \text{ or } \rho \frac{d\rho}{ds}.$$

13.* *A given curve possesses an infinity of involutes.*

Let P(x, y) be a point on a given curve and Q(ξ, η) a point on the tangent at P such that PQ$=-(s+\lambda)$, λ being constant.



The equation to the tangent at P is

$$\frac{X-x}{\cos\psi}=\frac{Y-y}{\sin\psi}=r$$

$\therefore \xi=x-(s+\lambda)\cos\psi$
$\eta=y-(s+\lambda)\sin\psi$

$$\frac{d\xi}{ds}=\frac{dx}{ds}-\cos\psi+(s+\lambda)\sin\psi\ \frac{d\psi}{ds}=(s+\lambda)\ \sin\ \psi\ \frac{d\psi}{ds}$$

and $\frac{d\eta}{ds}=\frac{dy}{ds}-\sin\psi-(s+\lambda)\cos\psi\ \frac{d\psi}{ds}=-(s+\lambda)\cos\psi\frac{d\psi}{ds}$

$$\therefore\quad \frac{d\eta}{d\xi}=-\cot\psi=-\frac{dx}{dy}.$$

$\therefore$ the tangent at Q to the Q-locus is parallel to the normal at P.

Hence QP is normal to the Q-locus at Q and consequently the given curve touches all the normals to the Q-locus, and every point of the curve is a point of contact with some normal. The given curve is therefore the evolute of the Q-locus. Hence the Q-locus is an involute of the given curve. But since λ is arbitrary, the given curve has an infinity of involutes.

Mechanical Description of an Involute.

Regard a given curve as a rigid curve. Let a thin inextensible, perfectly flexible thread be wound tightly round an arc OP of the curve and be stretched tightly along the tangent QQ_0. Then, as the thread is unwrapped, a point on it coincident with Q_0 will trace out the involute Q_0Q. For when the thread leaves the arc at P and the end is at Q along the tangent at P, we have $QP-Q_0O=$arc

OP and $\therefore$ Q is still on the involute through Q_0. Hence the end describes the involute Q_0 Q.

The names evolute and involute are given in consequence of this property.

Every point in OQ_0 (produced or not) describes an involute of the curve. The method of generating involutes shows that the involutes of a given curve cut the tangents at right angles, *i.e.*, they are orthogonal trajectories of these tangents. Hence any two of these involutes are curves having common normals such that the portions of the normals intercepted between them are equal. Curves thus treated are called parallel curves. Thus the involutes of a given curve are parallel curves.

Conversely, parallel curves are involutes of the same curve, for they have the same normals and, therefore, the same evolute.

Ex. 1. Find the evolute of the parabola $y^2=4ax$ and show that the length of the evolute from the cusp to the point in which the evolute intersects the parabola is $2a(3\sqrt{3}-1)$.

$$y=mx-2am-am^3 \qquad \ldots\ldots(1)$$

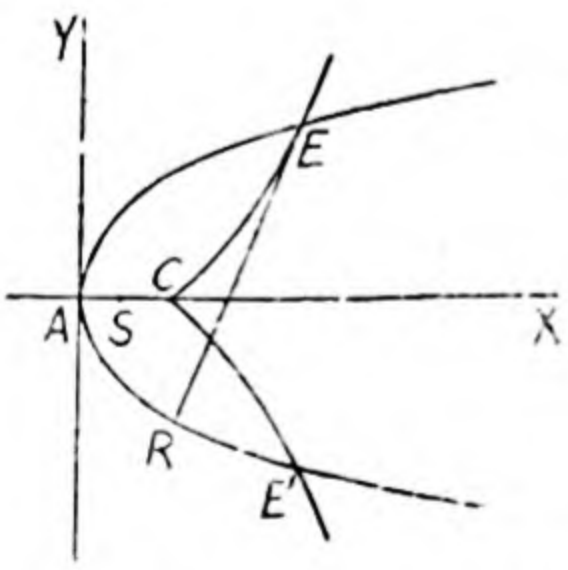
is normal to the parabola at a point $(am^2, -2m)$ for all values of m.

Its envelope is the evolute required.

$\therefore$ $$0=x-2a-3am^2 \qquad \ldots\ldots(2)$$

Eliminating m between (1) and (2) we get

$$27\,ay^2=4(x-2a)^3.$$



This equation has already been obtained for the locus of the centres of curvature of the parabola.

The cusp C corresponds to the vertex, the minimum point.

Let E and E′ be the points of intersection of the curves. Then the abscissa of E or E′ is given by

$$27\,a.\,4ax=4(x-2a)^3$$

or $x^3-6ax^2-15a^2x-8a^3=0$

the real root of which is $8a$.

Thus E is the point $(8a, 4\sqrt{2}a)$.

The abscissa of the point R whose centre of curvature is E is $\xi = am^2 = \frac{1}{3}(x-2a) = 2a$ by (2).

The radius of curvature at $R = \dfrac{2(SR)^{\frac{3}{2}}}{\sqrt{a}}$.

$$= \frac{2(\xi+a)^{\frac{3}{2}}}{\sqrt{a}} = 6\sqrt{3}\,a.$$

The radius of curvature at $A = \text{Lt}\ \dfrac{y^2}{2x} = 2a$.

$\therefore$ length of the arc CE = radius of curvature at R — radius of curvature at $A = 2a(3\sqrt{3}-1)$.

Ex. 2. Find the evolute of the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ and show that its length is $4\left(\dfrac{a^2}{b}-\dfrac{b^2}{a}\right)$.

The line $ax \sec\phi - by \operatorname{cosec}\phi = a^2-b^2$ .....(1)
is normal to the ellipse for all values of ϕ.

Differentiate (1) with respect to ϕ.

$ax \sec\phi \tan\phi + by \operatorname{cosec}\phi \cot\phi = 0$ .....(2)

Eliminate ϕ between (1) and (2).

From (2) $\dfrac{ax}{\cos^3\phi} = \dfrac{-by}{\sin^3\phi}$

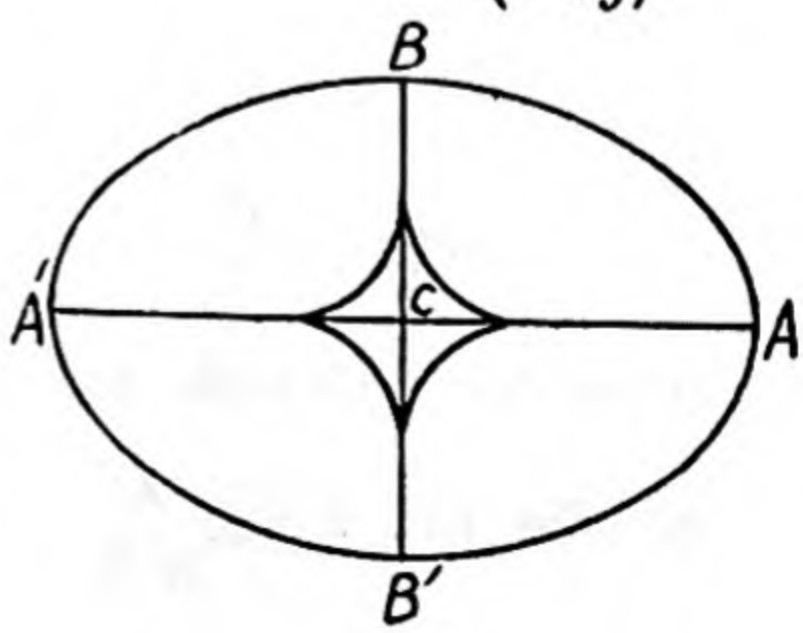
i.e., $\dfrac{(ax)^{\frac{1}{3}}}{\cos\phi} = \dfrac{(-by)^{\frac{1}{3}}}{\sin\phi} = \dfrac{\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{1}{2}}}{\sqrt{(\cos^2\phi+\sin^2\phi)}}$

$\therefore$ $\sec\phi = \dfrac{\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{1}{2}}}{(ax)^{\frac{1}{3}}}$, $\operatorname{cosec}\phi = \dfrac{\left[(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}\right]^{\frac{1}{2}}}{(-by)^{\frac{1}{3}}}$

Substituting in (1) we get

$$(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}} = (a^2-b^2)^{\frac{2}{3}}.$$

The evolute has four cusps on the axes corresponding to the vertices, the maximum and minimum points.



The radius of curvature at any point of the ellipse $= \dfrac{a^2b^2}{p^3}$.

The radius of curvature at $A' = \frac{a^2b^2}{a^3}$ [$\because$ $p^3 = a^3$ at A']

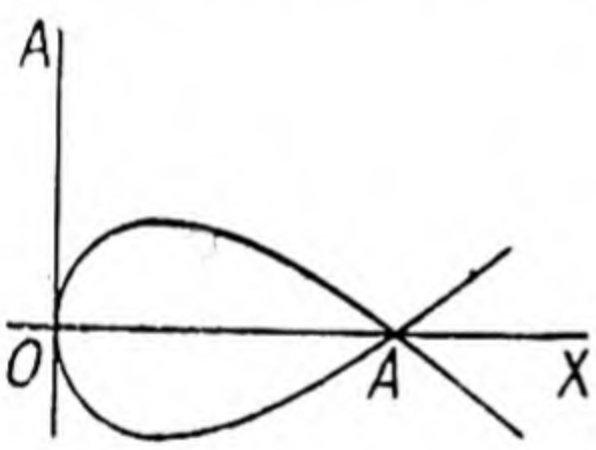
$$= \frac{b^2}{a}$$

and that at $B' = \frac{a^2b^2}{b^3} = \frac{a^2}{b}$. [$\because$ $p^3 = b^2$ at B']

Since the curve is symmetrical about the axes, the length of the evolute is therefore

$$4\left(\frac{a^2}{b} - \frac{b^2}{a}\right).$$

Ex. 3. Find the length of the evolute corresponding to the loop of the curve $9ay^2 = x(x-3a)^2$.



The curve is symmetrical about the x-axis.

The x-axis cuts it in points (0, 0), (3a, 0).

A (3a, 0) is a node on the curve.

The required length $= 2(\rho_A - \rho_0)$.

Since the y-axis touches the curve at the origin.

$$\therefore \quad \rho_0 = \text{Lt } \frac{y^2}{2x} = \text{Lt } \frac{(x-3a)^2}{18a} = \frac{a}{2}.$$

Shift the origin to A and the equation takes the form $9ay^2 = (x+3a)x^2$.

Put $y = px + \frac{qx^2}{2} + \cdots\cdots$ and we have

$$9a\left(px + \frac{qx^2}{2} + \cdots\cdots\right)^2 = x^2(x+8a).$$

Comparing the co-efficients :

$9ap^2 = 3a$, *i.e.*, $p = \pm\frac{1}{\sqrt{3}}$; $9apq = 1$, *i.e.*, $q = \pm\frac{1}{3\sqrt{3a}}$.

$$\therefore\ \rho_A = \frac{(1+p^2)^{\frac{3}{2}}}{q} = \frac{\left(\frac{4}{3}\right)^{\frac{3}{2}}}{\frac{1}{3\sqrt{3}a}} = 8a.$$

$\therefore$ The required length $= 15a$.

EXAMPLES XLVI

1. Find the evolute of the hyperbola $x = a \cosh u$, $y = b \sinh u$.

2. Show the at centre of curvature of the curve

$x = at^2$, $y = at^3$ is $\left(-at^2 - \frac{9}{2}at^4,\ 4at^3 + \frac{4}{3}at\right)$ at t.

Deduce the equation of the evolute and prove that the radius of curvature of the evolute at the origin is $\frac{8}{9}a$.

3. Show that the equation of the evolute of the rectangular hyperbola $xy = c^2$ is $(x+y)^{\frac{2}{3}} - (x-y)^{\frac{2}{3}} = (4c)^{\frac{2}{3}}$.

4. Show by finding the equation of the tangent to the evolute of the parabola $y^2 = 4ax$, that it is identical with the equation of the normal to the parabola.

5. Show that the pedal equation of the involute of a circle (radius a) is $r^2 = p^2 + a^2$ and the intrinsic equation is $s = \frac{a}{2}\psi^2$.

6. Show that the evolute of the curve $r^2 - a^2 = mp^2$ has for its equation $r^2 - (1-m)a^2 = mp^2$.

7. Prove that the pedal equation of the four cusped hypocycloid $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ is $r^2 + 3p^2 = 16a^2$.

Hence or otherwise show that its evolute is the hypocycloid $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$.

8. Prove that the centre of curvature at any point of the equiangular spiral $r = ae^{\lambda\vartheta}$ is the extremity of the polar subnormal.

Hence show that the first evolute and all subsequent evolutes are equiangular spirals.

Find the equation which λ must satisfy if the given curve be its own evolute.

9. Prove that the evolute of the cardioid $r=a(1-\cos\theta)$ is the cardioid $r=\frac{a}{3}(1-\cos\theta)$.

10. If a curve passes through the origin, show that its radius of curvature at the origin is $\frac{1}{2}$ Lt $\frac{x^2+y^2}{x\sin\alpha-y\cos\alpha}$ where $y=x\tan\alpha$ is the tangent at the origin.

Hence show that radius of curvature of the curve

$x^4+y^2=2a(x+y)$ at the origin is $2a\sqrt{2}$.

11. The radius of curvature of a plane curve always subtends a right-angle at the origin. Show that the curve cuts the radius vector at a constant angle.

12. Prove that the equation of the normal to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ may be written in the form $y\cos\phi-x\sin\phi=a\cos 2\phi$.

Hence show that the evolute of the curve is

$$(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}.$$

13. Prove that the envelope of the common chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and its circle of curvature is the curve

$$\left(\frac{x}{a}+\frac{y}{b}\right)^{\frac{2}{3}}+\left(\frac{x}{a}-\frac{y}{b}\right)^{\frac{2}{3}}=2.$$

14. Show that an ellipse is the envelope of its circles of curvature.

15. If ρ_1, ρ_2 be the radii of curvature at the extremities of two conjugate diameters of an ellipse (semi-axes a, b), prove that

$$\rho_1^{\frac{2}{3}}+\rho_2^{\frac{2}{3}}=\left(\frac{a^2}{b}\right)^{\frac{2}{3}}+\left(\frac{b^2}{a}\right)^{\frac{2}{3}}.$$ [See Ex. 3 page 323.]

16. If r, ρ be the radius vector and radius of curvature of a curve and r', ρ' the corresponding quantities of its inverse, prove that

$$\frac{r}{\rho}\cdot\frac{r'}{\rho'}=2\sin\phi.$$

17. If ρ' be the radius of curvature of the first positive pedal of the curve $r=f(p)$, show that $\frac{1}{\rho'}=\frac{1}{r}+\frac{d}{dp}\left(\frac{p}{r}\right)$.

18. Find the length of the evolute corresponding to the loop of the curve $y^2+4y=x^3+6x^2+9x$.

19. Prove that the locus of the focus of a parabola which has a contact of the 2nd order with a given curve at a given point is a circle.

20. A tangent to the evolute of a parabola at the point where it meets the parabola is also a normal to the evolute at the point where it again meets the evolute.

21. If ρ and ρ' be the radii of curvature at corresponding points of curve and its evolute, prove that

$$\frac{\rho'}{\rho}=\{3y'y''^2-y'''(1+y'^2)\}y''^2.$$

22. If ρ and ρ' be the radii of curvature of the curve $r=f(p)$ and its reciprocal polar at the corresponding points, prove that $\rho'=\frac{k^2r^3}{\rho p^3}$ where k^2 is the constant of reciprocation.

23. Show that the radius of curvature of the envelope of the line $x\cos\alpha+y\sin\alpha=f(\alpha)$ is $f(\alpha)+f''(\alpha)$.

24. Show that the radius of curvature ρ' of the inverse with regard to the origin of the curve $r=f(p)$ is $\rho'=\frac{k^2\rho}{2p\rho-r^2}$, where ρ and ρ' are radii of curvature at the corresponding points of the curve and its inverse.

25. If ρ be the radius of curvature of the curve $r^m=a^m\sin m\vartheta$ at the point whose distance measured along the curve

from a fixed point is s, prove that

$$(m-1)(m+1)^2\rho\ \frac{d^2\rho}{ds^2}-m(m+1)^2\left(\frac{d\rho}{ds}\right)^2=m(m-1)^2.$$

26. For a curve defined by the equation $x=f(\psi)$, prove that the projection of the radius vector on the tangent is $\frac{dp}{d\psi}$ and the radius of curvature is $p+\frac{d^2p}{d\psi^2}$.

27. Find the radius and the centre of curvature of a curve given by $x=\phi(t)$, $y=f(t)$.

28. If the circle of curvature at the point of the parabola $y^3=4ax$ touches the directrix, prove that the point lies at the intersection of the parabola with the line $4x-5a=0$.

29. If the circle of curvature at a point of the hyperbola $4xy=a^2+b^2$ touches one of the asymptotes, prove that the point lies at the intersection of the hyperbola with one of the curves,

$$3x^4-2(a^2-b^2)x^2+6x^2y^2-y^4=0$$
$$3y^4-2(a^2-b^2)y^2+6x^2y^2-x^4=0.$$

CHAPTER XIV

CURVE TRACING

1. In the present chapter we shall show how the results obtained in the preceding chapters can be used for tracing curves from their Cartesian or Polar equations ; for this purpose we shall lay down certain rules which are quite sufficient for tracing most of the curves that usually occur. Also we shall give the principal properties of some of the well-known curves.

It may be remarked that if we tabulate the corresponding values of x and y which satisfy $f(x, y)=0$ or of r and θ which satisfy $\phi(r, \theta)=0$, we obtain a group of points which lie on the curve ; but it is only in every particular cases that we are able to draw the corresponding graphs completely.

Rules for Cartesian Equations.

Symmetry.

(*i*) If the equation of the curve remains unchanged when x is changed into $-x$, then there is symmetry about the y-axis. In the case of algebraic curves this implies that only even powers of x occur in its equation.

For example $x^2=4ay$ and $y=\cos x$ are symmetrical with respect to the y-axis.

(*ii*) If the equation of the curve remains unchanged when y is changed into $-y$, then there is symmetry about the x-axis.

For example $y^2=4ax$ is symmetrical about the x-axis while $x^2/a^2+y^2/b^2=1$ is symmetrical about both the axes.

(*iii*) If the equation of the curve remains unchanged when both x and y are changed into $-x$ and $-y$, then there is symmetry in opposite quadrants.

For example, $y=\sin x$ and $xy=c^2$ have got symmetry in opposite quadrants.

It may be noticed that if a curve is symmetrical with res-

pect to both the axes then it also possesses symmetry in opposite quadrants. But the converse is not true.

(*iv*) If the equation of the curve remains unchanged when x and y are interchanged, then the curve is symmetrical about the line $y=x$. Find in this case where the curve cuts this line.

It may be remarked that sometimes an obvious transformation introduces symmetry in the curve.

For example $\frac{(x-d)^2}{a^2}+\frac{(y-c)^2}{b^2}=1$ does not possess any of the above symmetries but if we shift the origin to the point (d, c) the equation becomes $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, which is symmetrical about both the new axes.

II. Notice whether the curve passes through the origin, and if it does, find the tangent or tangents at that point. In case there are more than one tangents, find the nature of the singularity at the origin.

III. Find the points where the curve crosses the co-ordinate axes and the tangents at those points.

If necessary, as is sometimes the case, find a few other particular points which obviously lie on the curve.

IV. Find the asymptotes, if any, and determine if, necessary from which side the curve approaches any particular asymptote.

V. Find $\frac{dy}{dx}$, if convenient, and see where it is zero or infinite.

VI. Find points of inflexion, and other singular points if any, and in the case of the latter examine their nature.

VII. Find, if possible, the region or regions in which the curve does not exist. This is usually done by solving for x and for y separately. Consider both positive as well as negative values of x and y.

For example, in the curve $y^2(2a-x)=x^3$ we have $y^2=\frac{x^3}{2a-x}$. If x is negative, then y^2 is negative so that y is imaginary:

no part of the curve therefore lies to the left of the y-axis. Also when x is positive, it must be less than $2a$. The curve, therefore, lies entirely between the lines $x=0$ and $x=2a$.

In this connection it may also be sometimes useful to put the equation of the curve in the form $x=f(t)$, $y=\phi(t)$ when possible. For example, the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}$ can be put in the form $x=a\cos^3 t$, $y=a\sin^3 t$, which shows that the curve lies wholly within the circle $x^2+y^2=a^2$.

Ex. 1. Trace the curve $y=x^3$.

I. (*a*) If x is changed into $-x$, the equation changes, so that there is no symmetry about the y-axis.

(*b*) There is no symmetry about the x-axis.

(*c*) If x is changed into $-x$ and y into $-y$, the equation remains unchanged; so that there is symmetry in opposite quadrants.

(*d*) If x and y are interchanged, the equation changes; therefore there is no symmetry about the line $y=x$.

II. The curve passes through the origin, tangent thereat being $y=0$.

III. The curve crosses the axes only at the origin.

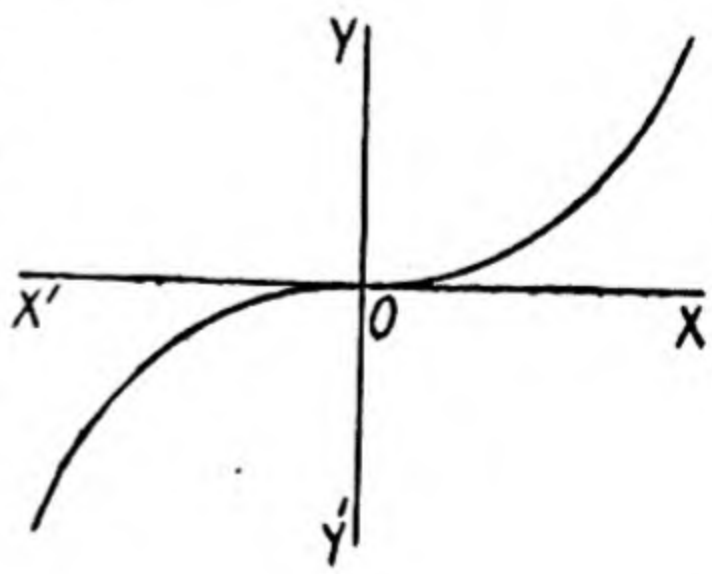
IV. The curve has no asymptotes.

V. $\frac{dy}{dx}=3x^2$ and vanishes at the origin..

VI. $\frac{d^2y}{dx^2}=6x$ which vanishes when $x=0$ and changes sign. The origin therefore is a point of inflexion.

VII. $y=x^3$, x and y are both positive or both negative; also as x increases numerically, so does y and there is no limit to this increase. The curve therefore lies in the first and the third quadrant and extends to infinity.

The sketch of the curve can now be drawn.



This is known as **cubical parabola.**

Ex. 2. Trace the curve $y^2=x^3$.

I. (*a*) There is no symmetry about the y-axis.

(*b*) There is symmetry about the x-axis.

(*c*) There is no symmetry in opposite quadrants.

(*d*) Thers is no symmetry about the line $y=x$.

II. The curve passes through the origin, tangents thereat being $y=0$ and $y=0$; the origin therefore is a cusp. Also the two values of y are of opposite sign ; so that the cusp is of the first species.

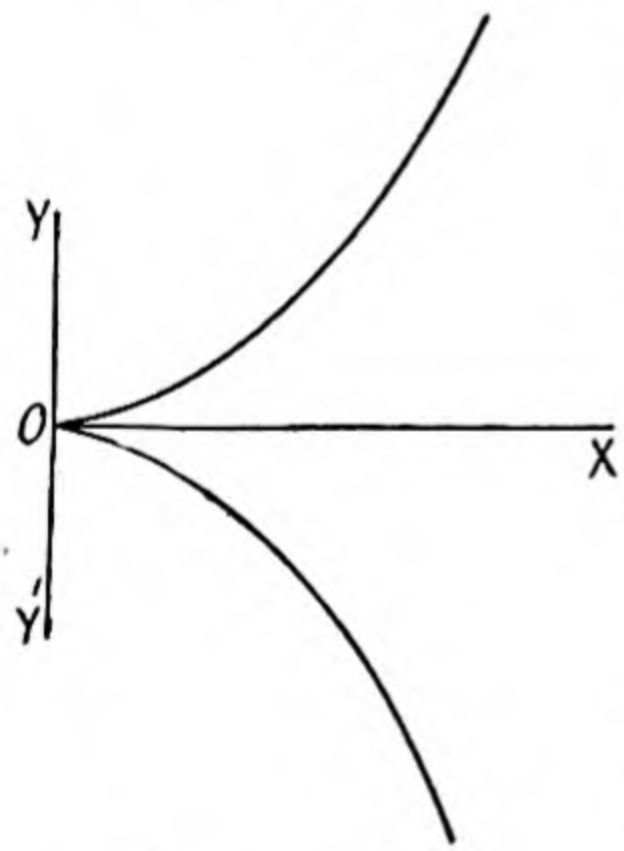
III. The curve crosses the axes only at the origin.

IV. The curve has no asymptotes.

V. $\frac{dy}{dx}=\frac{3}{2}x^{\frac{1}{2}}$ which vanishes at the origin.

VI. $\frac{d^2y}{dx^2}=\frac{3}{4}x^{-\frac{1}{2}}$ and therefore there is no point of inflexion.

VII. $y^2=x^2$. If x is negative, y is imaginary ; so that x is not negative. The curve does not lie to the left of the y-axis. When x is positive and increases so does y and there is no limit to the increase ; the curve therefore extends to infinity.



This is known as **semicubical parabola.**

It may be remarked that the envelope of normals of the parabola $y^2=4ax$ is the semicubical parabola.

$$27ay^2=4(x-2a)^3$$

Ex. 3. Trace the curve $x^3+y^3=3axy$.

I. There is symmetry about the line $x=y$ and no other

symmetry. The line $x=y$ cuts the curve at the points (0, 0), (0, 0) and $(3a/2, 3a/2)$.

II. The curve goes through the origin, tangents thereat being $xy=0$, *i.e.* $x=0$ and $y=0$; the origin is therefore a node.

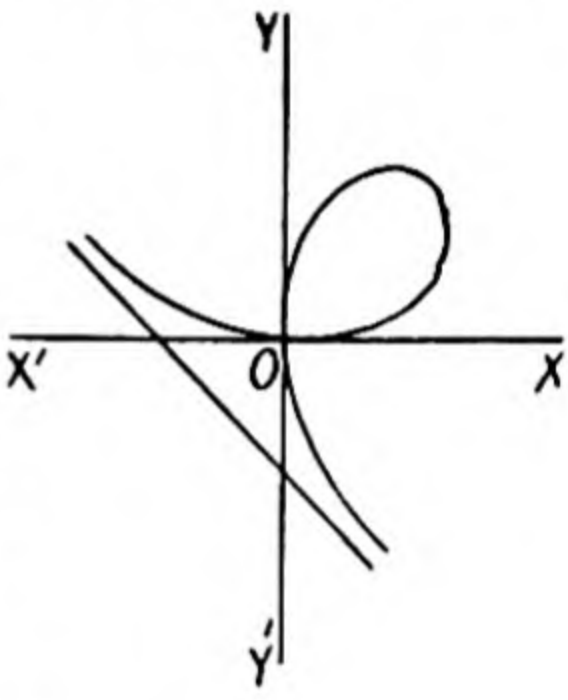
III. The curve crosses each of the axes at (0, 0).

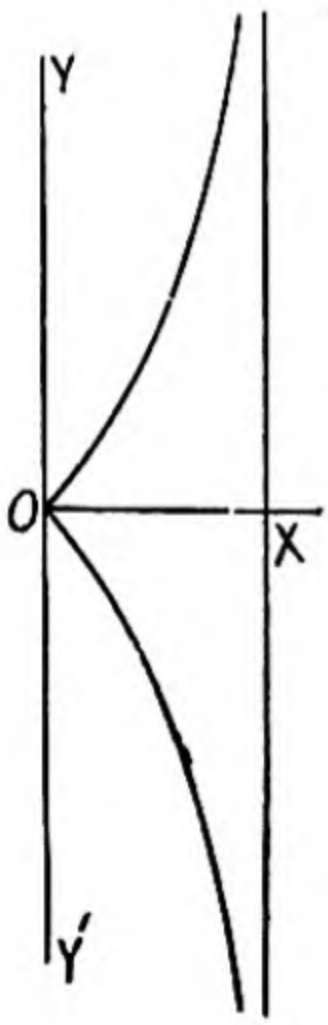
IV. $x+y+a=0$ is an asymptote and the curve approaches the asymptote from above in the second and the fourth quadrants.

V. $\frac{dy}{dx}$ is equal to 0 as well as infinity at the origin.

VI. It has no points of inflexion.

VII. We cannot solve for x or y conveniently. But from the equation of the curve it follows that x and y cannot both be negative. The curve therefore does not lie in the third quadrant.





The shape of the curve can now be drawn. This is known as **Folium of Descartes.**

Ex. 4. Trace the curve $y^2(2a-x)=x^3$.

I. There is symmetry about the axis of x.

II. The curve passes through the origin, tangents as the origin being $y=0$, $y=0$; origin is therefore a cusp.

III. The curve crosses the axes only at the origin.

IV. $x=2a$ is an asymptote.

V. $\frac{dy}{dx}$ is zero at the origin.

VI. There is no point of inflexion and no other singular point except the origin.

VII. $y^2=x^3/(2a-x)$. If x is negative, y^2 is negative so that no part of the curve lies on the left side of the y axis. Also when x is positive, it is not greater than $2a$.

The curve can now be easily traced.

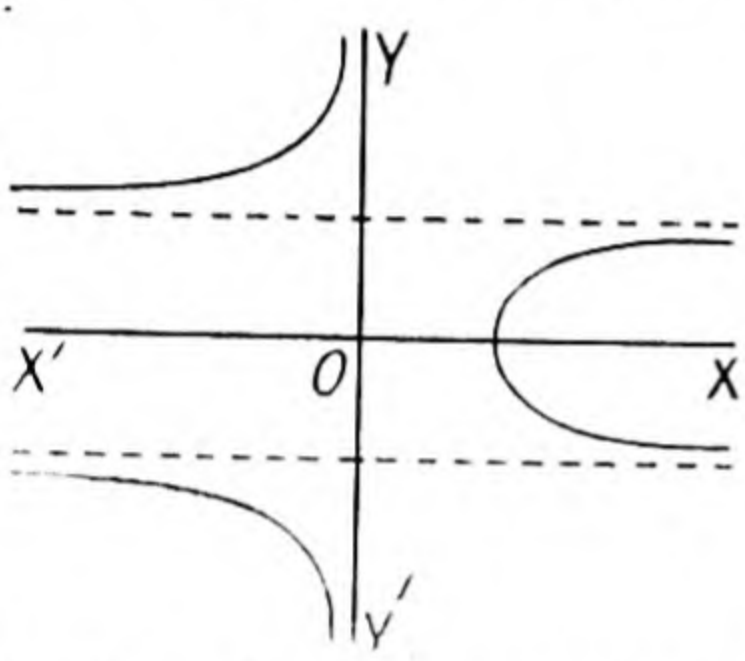
It is called **Cissoid of Diocles.** The duplication of a cube can be effected by means of this curve.

Ex. 5. Trace the curve $y^2x=a^2(x-a)$.

I. There is symmetry about the x-axis and no other symmetry.

II. The curve does not pass through the origin.

III. The curve cuts the x-axis at the point $(a, 0)$ and the y-axis at an infinite distance.



IV. $x=0$, $y=\pm a$ are asymptotes.

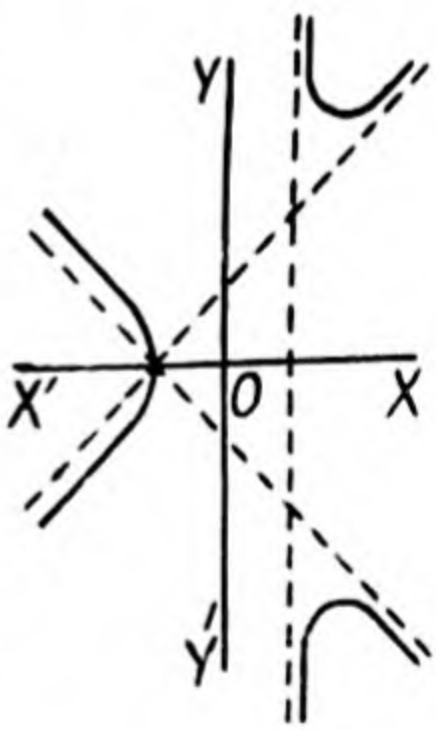
V. $\dfrac{dy}{dx}$ is infinite at $(a, 0)$.

VI. It has no point of inflexion and no singular points.

VII. $y^2=a^2(x-a)/x=a^2-a^3/x$; when x is positive, it must be greater than a, in which case y is less than a numerically, and when x is negative there is no restriction on it.

Also $x=a^3/(a^2-y^2)$. When y is numerically less than a, x is positive otherwise x is negative.

Ex. 6. Trace the curve $y^2(x-a)=x^2(x+a)$.



I. There is symmetry about the x-axis and no other symmetry.

II. The curve goes through the origin, the tangents there-at being $x^2+y^2=0$, which are imaginary so that origin is an isolated point on the curve.

III. The curve cuts the x-axis at $(-a, 0)$ and does not cut the y-axis. Also when $x\to a$, y is infinite.

IV. The lines $y=\pm(x+a)$, $x=a$ are asymptotes.

V. $\dfrac{dy}{dx}$ is infinite at $(-a, 0)$.

VI. No points of inflexion and no singular points.

VII. $y^2=x^2(x+a)/(x-a)$.

When x is positive it must be greater than a ; when x is negative it must be greater than a numerically.

Ex. 7. Trace the curve $a^2y^2=x^2(2a-x)\ (x-a)$.

I. There is symmetry about the x-axis and no other symmetry.

II. The curve passes through the origin, tangents there-at being imaginary, $y^2+2x^2=0$. The origin is therefore an isolated point.

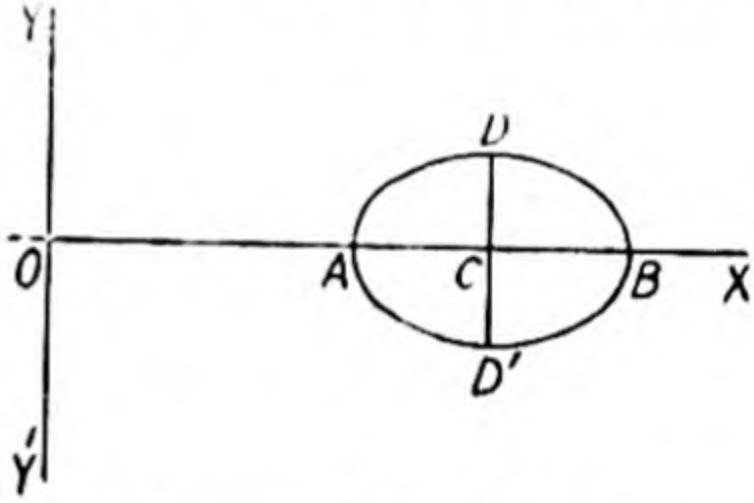
III. When $x=0$, $y=0$; and when $y=0$, $x=0$, a and $2a$.

IV. No asymptotes.

V. $\frac{dy}{dx}$ is infinite at $x=a$ and $2a$.

VI. No point of inflexion and no singular points.

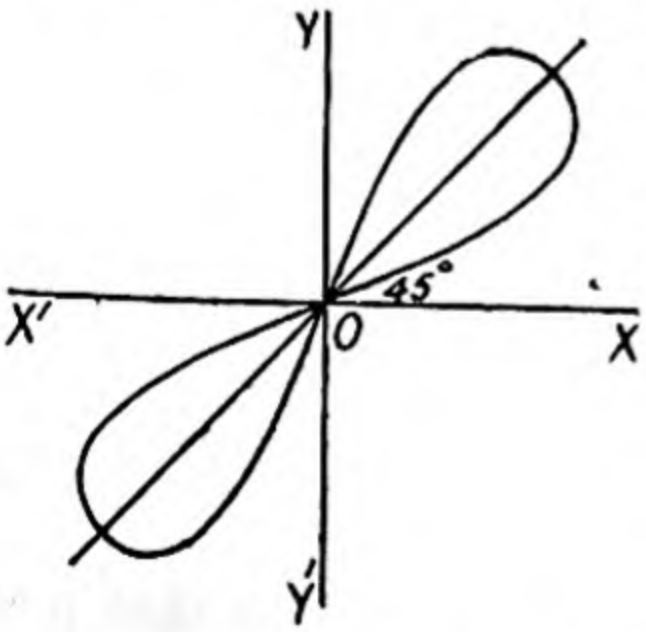
VII. $a^2y^2=x^2(x-a)(2a-x)$.



When x is positive, it must be greater than a and less than $2a$; when x is negative y^2 is negative so that x is not negative.

Ex. 8. Trace the curve $x^4+y^4=4a^2xy$.

There is symmetry about the line $x=y$ as well as in opposite quadrants. The curve cuts the line $y=x$ at the points $(0, 0)$, $(\sqrt{2}a, \sqrt{2}\ a)$ and $(-\sqrt{2}a, -\sqrt{2}a)$.



The curve goes through the origin, the tangents at the origin being $x=0$ and $y=0$; the origin is a node.

The curve crosses the axes at the origin.

There are no asymptotes. No singular points except the origin.

Also $a^2xy=x^4+y^4$ indicates that x and y are both positive or both negative so that the curve lies only in the first and the third quadrants.

GIRLS COL

3. **Rules for Polar Co-ordinates.** The rules of the previous section are also applicable to the case of polar co ordinates, with slight and corresponding modifications.

For example, there is symmetry about the initial line if the equation remains unaltered when θ is changed into $-\theta$; also there is symmetry about the origin if the equation remains unchanged when r is changed into $-r$. Again if r is regarded as essentially positive, the curve is non-existent in the region bounded by the lines $\theta=\theta_1$ and $\theta=\theta_2$ if r is negative or imaginary when θ lies between θ_1 and θ_2.

It may be remarked that sometimes it is more convenient to change Polar co-ordinates to Cartesian co-ordinates or conversely.

The following examples will illustrate the points more fully :

Ex. 1. Trace the curve $r=a(1+\cos\theta)$.

The following facts hold true :

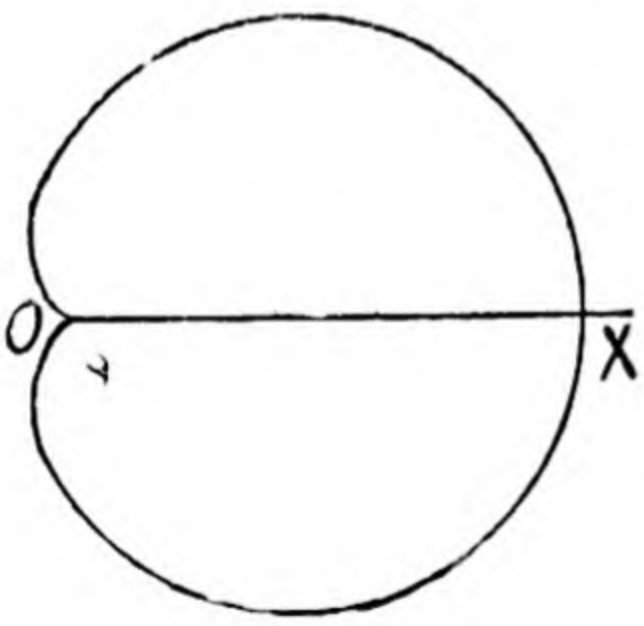
I. There is symmetry about the initial line.

II. The curve passes through the origin, tangent at that point being the initial line $\theta=0$.

III. The curve crosses the initial line at $(2a, 0)$.

IV. There are no asymptotes.

V. The curve has no points of inflexion.



VI. r is never greater than $2a$. As θ increases from 0 to π, r decreases from $2a$ to 0.

VII. When $\theta=0$, $\frac{\pi}{2}$ and π, $r=2a$, a, 0 respectively.

The curve can now be easily traced.

This is known as a **cardioid.**

Ex. 2. Trace the curve $r^2=a^2\cos 2\theta$.

I. There is symmetry about the initial line as well as the origin or pole.

II. The curve passes through the pole, tangents at that point being $\theta=\pm\frac{\pi}{4}$.

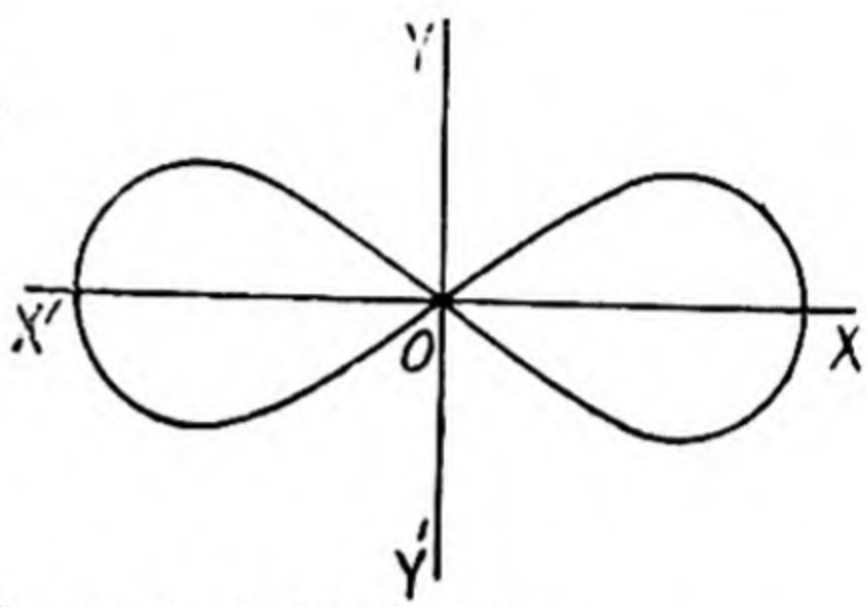
III. The curve crosses the initial line at the points $(\pm a, 0)$.

IV. No asymptotes.

V. No points of inflexion.

VI. r is never greater than a so that the curve lies wholly within the circle $r=a$. Also the curve is non-existent in the intervals $(\pi/4,\ 3\pi/4)$ and $(5\pi/4,\ 7\pi/4)$ because r^2 becomes negative.

VII. When $\theta=0,\ \pi/4,\ 3\pi/4,\ \pi,\ 5\pi/4,\ 7\pi/4$ and 2π, $r=a,\ 0,0,\ a.\ 0,0,\ a$ respectively.



The curve can now be easily traced.

This is known as **Lemniscate of Bornoulli.**

Ex. 3. Trace the curve $r=a \sin 3\theta$.

I. There is no symmetry about the initial line.

II. The curve passes through the origin, the tangents at the origin being $\theta=0$ and $\theta=\pm\pi/3$.

III. The curve crosses the initial line through the pole only.

IV. There is no asymptote.

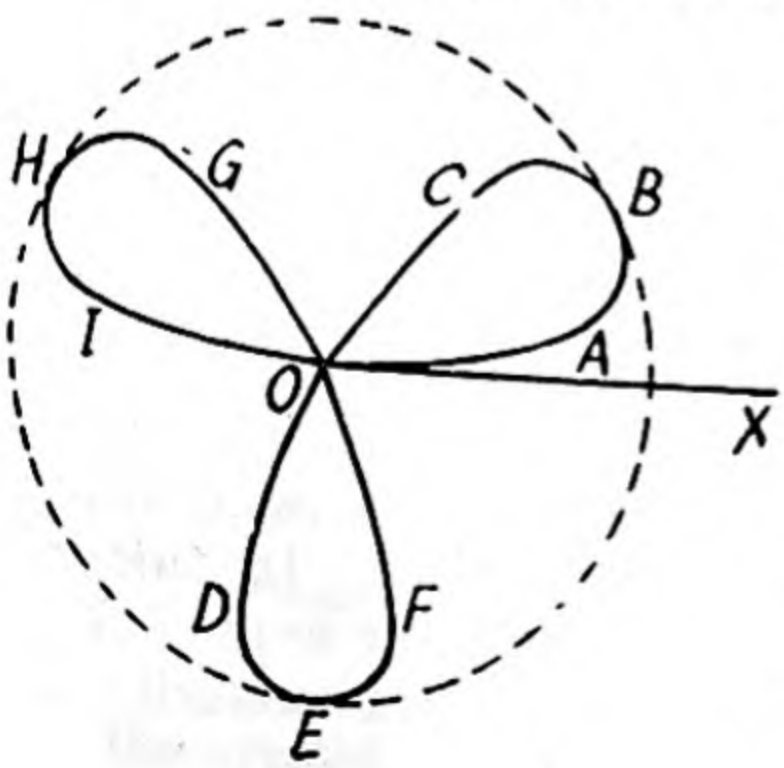
V. There is no point of inflexion.

VI. No other singular point except the origin.

VII. $\sin 3\theta=r/a$; r is never greater than a, so that the curve lies wholly within the circle $r=a$.

As θ increases from 0 to $\pi/6$, r increases from 0 to a ; as θ increases from $\pi/6$ to $\pi/3$, r decreases from a to 0.

Further, as θ increases from $\pi/3$ to $\pi/2$, r is negative and numerically increases from 0 to a ; as θ increases from $\pi/2$ to $2\pi/3$, r increases from $-a$ to 0. Further, as θ increases from $2\pi/3$ to $5\pi/6$, r increases from 0 to a and as θ increases from $5\pi/6$ to π, r decreases from a to 0.



As θ increases further we get no new points on the curve.

The curve consists of three loops described in the order OABCO, ODEFO and OGHIO.

It may be remarked that the curve $r=a \sin (2n+1)\theta$ is of a similar shape consisting of $2n+1$ loops, and the curve $r=a \sin 2n\theta$ consists of $4n$ similar loops.

Ex. 4. Trace the curve $r=b \cos \theta \pm a$.

I. There is symmetry about the initial line and no other symmetry.

II. The curve passes through the origin. If we convert to Cartesian co-ordinates, the equation becomes

$$(x^2+y^2-bx)^2=a^2(x^2+y^2).$$

This shows that the curve passes through the origin, the tangents at the origin being $a^2y^2+(a^2-b^2)x^2=0$, which are imaginary if $a>b$, real and distinct if $a<b$ and coincident if $a=b$. Hence the origin is a conjugate point, a node or a cusp according as (*i*) $a>b$ (*ii*) $a<b$ (*iii*) $a=b$. We shall take up these three cases separately ; but it may be observed that in all the three cases r is never greater than $b+a$ and therefore the curve is a closed one.

(*i*) $a>b$. Let us first take the upper sign, that is $r=b \cos \theta+a$.

When $\theta=0$, $r=a+b$ and as θ increases r decreases till when $\theta=\pi$, $r=a-b$ which is its minimum value.

We get the same curve when we take the lower sign, that is, $r=b \cos \theta-a$.

(*ii*) $a<b$. In this case we get two different branches according as we take $r=b \cos \theta+a$ (*ii*) (*a*)

or $r=b \cos \theta-a$ (*ii*) (*b*)

But if we admit negative values of r, both the equations $r=b \cos \theta \pm a$ give the same curve which will now consist of two parts, one which is a loop within the other. In this case there is, as already noted, a deuble point at the origin. When negative values of r are admitted, then θ will necessarily vary from 0 to π and not 0 to 2π.

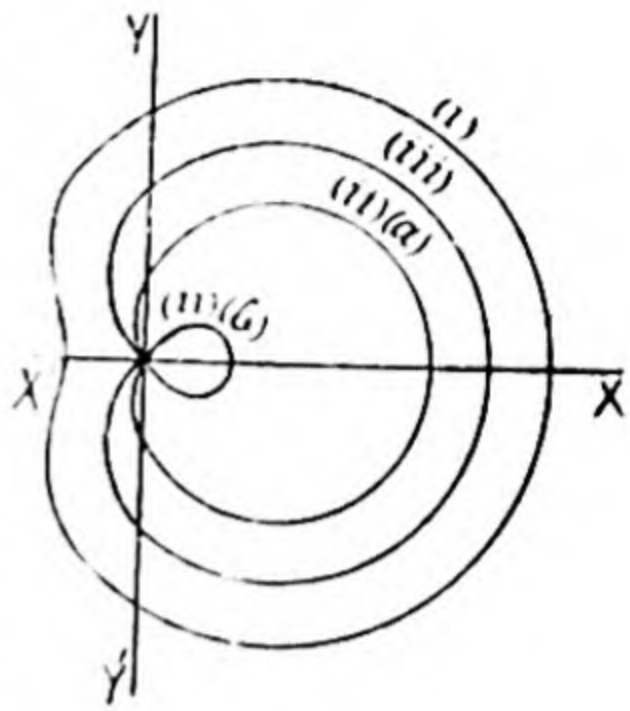
(*iii*) $a=b$. This can be traced as in example 1 above. We get only one curve from both the equations

$$r=a\ (\cos\theta\pm 1).$$

The particular form of the Limacon $r=b\cos\theta+a$, is the Trisectrix

$$r=a(2\cos\theta+1)$$

by means of which a given angle can be trisected.

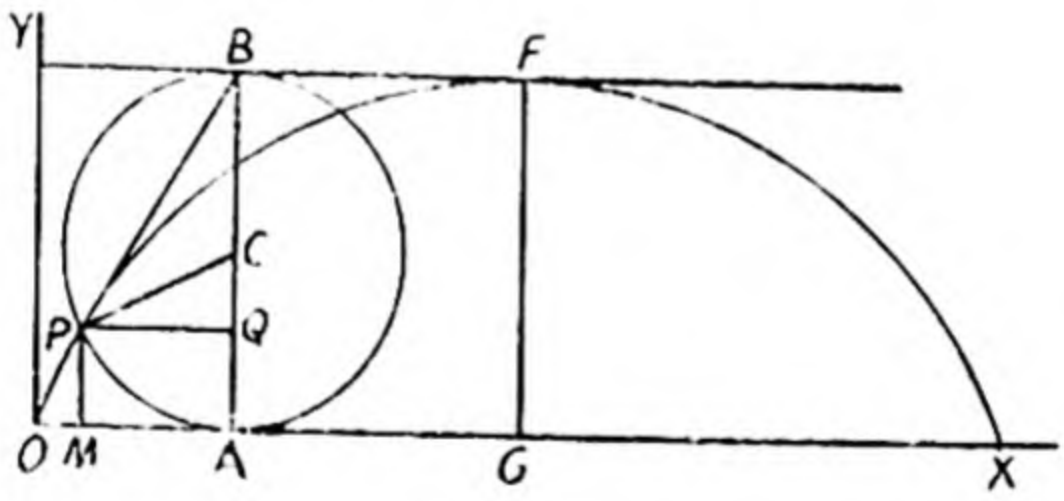


We shall now give a very brief account of some of the well known curves.

Some Well-known Curves

The Cycloid. The cycloid is described by a point P on the circumference of a circle of radius a which rolls without sliding on a fixed straight line OX. The fixed straight line is called the *base*.

Take this line as the axis of x and for axis of y the line drawn perpendicular to it at the point where at some instant P is in contact with the line OX.



Consider any other position of the generating circle. From the definition of rolling, it follows that OA is equal to the arc AP. Let θ be the angle between CA and CP. We shall express x and y in terms of θ. We have $OA=AP=a\theta$; therefore

$$x=OA-MA=OA-PQ=a(\theta-\sin\theta).$$

$$y=AC-QC=a-a\cos\theta=a\ (1-\cos\theta).$$

These two equations furnish a parametric representation of the cycloid.

It is obvious that the cycloid is composed of an unlimited succession of arches situated above OX, having $2a$ for their height and $2\pi a$ for their base.

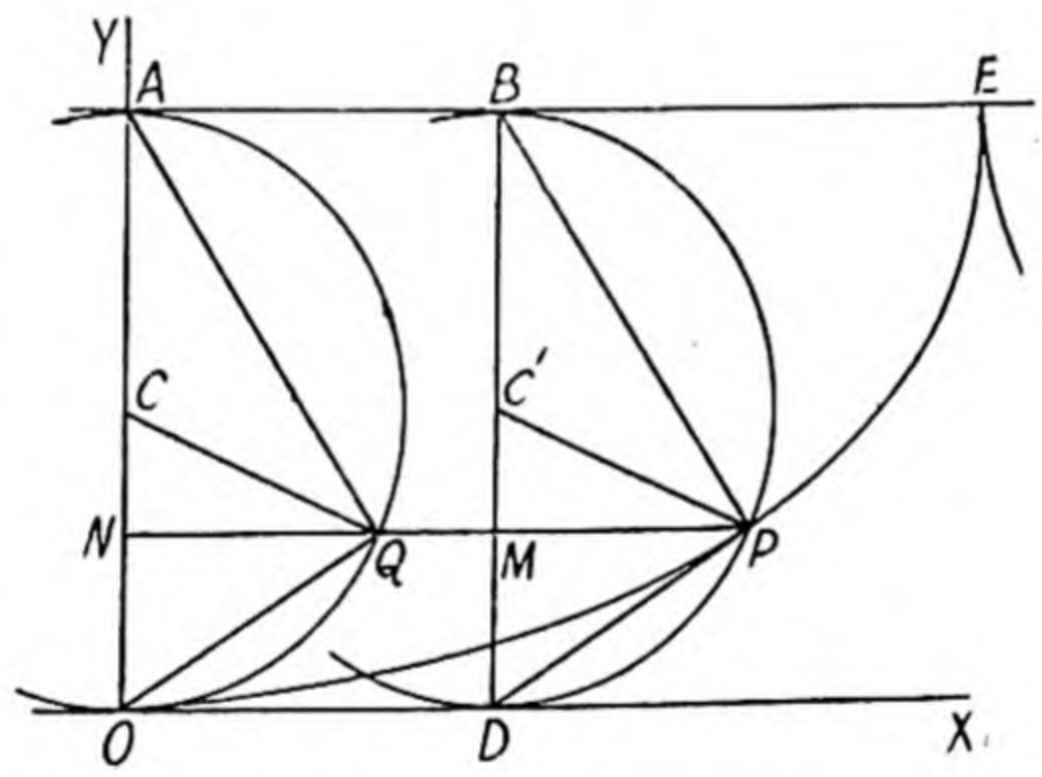
$$\frac{dx}{d\theta} = a(1-\cos\theta) = y,$$

and $\frac{dy}{d\theta} = a \sin\theta$; so that $\frac{dy}{dx} = \frac{a \sin\theta}{y}$.

Therefore subnormal $= a \sin\theta =$ PQ or MA.

Therefore it follows that the normal at P passes through A, the point of contact of the generating circle with x-axis. The point A is called the *instantaneous centre of rotation.*

Points like F are called *vertices* and lines like FG called its *axes.*



Sometimes the equation of the cycloid is put in a slightly different form as follows :

Let AB be the line on which the circle rolls. It is convenient to use the circle described on axis AO as diameter as the circle of reference. Let BPD be any other position of the rolling circle, B the point of contact with the base, C′ the centre, D the extremity of the diameter through B and let P be the position of the tracing point. Draw PMN parallel to the base, meeting DB and OA in M and N respectively and the circle of reference in Q. If OD and OA be taken as axis of x and y, the co-ordinates of P will be given by

$$x = \text{NP} = \text{AB} + \text{MP},\ y = \text{ON} = \text{DC}' - \text{MC}'.$$

If a be the radius of the rolling circle and θ the angle PC'D through which it turns as the tracing point P travels from O to P, then we have $AB=a\,\theta$ and $PM=a\sin\theta$, $C'M=a\cos\theta$ and therefore

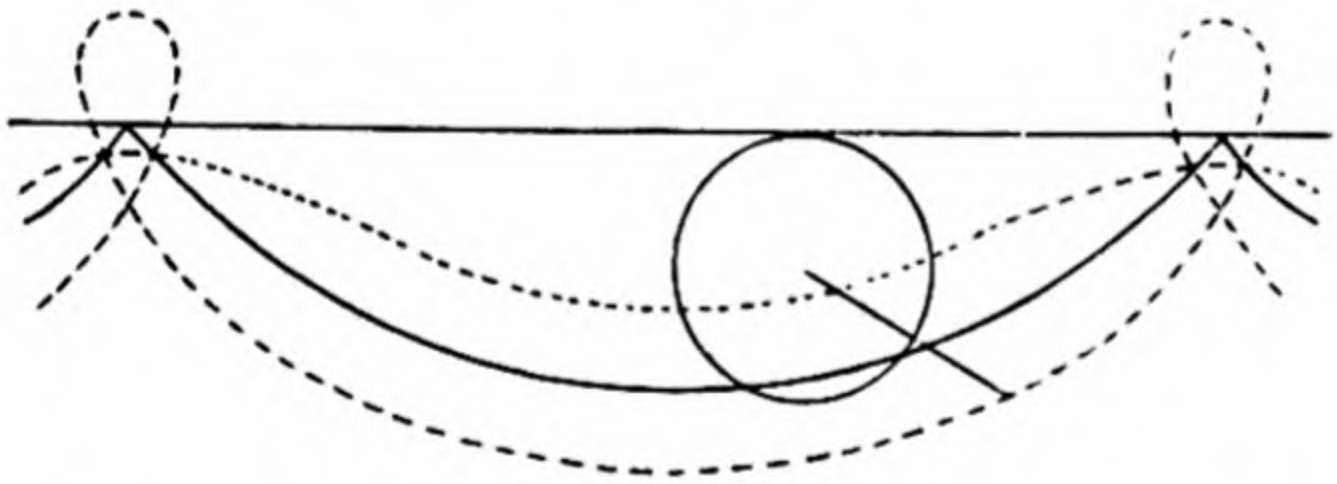
$$x=a\,(\theta+\sin\theta).$$
$$y=a\,(1-\cos\theta).$$

The curve traced out by a point of the rolling circle which is not on the circumference is called a *trochoid*.

If in the above figure the tracing point be in the radius CP at a distance h from the centre, its co-ordinates x and y are given by the equations:

$$x=a\,\theta+h\sin\theta$$
$$y=a-h\cos\theta.$$

When $h>a$ the curve has got loops, which in the particular case $h=a$ (of the cycloid) degenerate into cusps. When $h<a$ the curve does not meet the base. The figure shows the three cases when $h=a/2$, $h=a$ and $h=3a/2$.



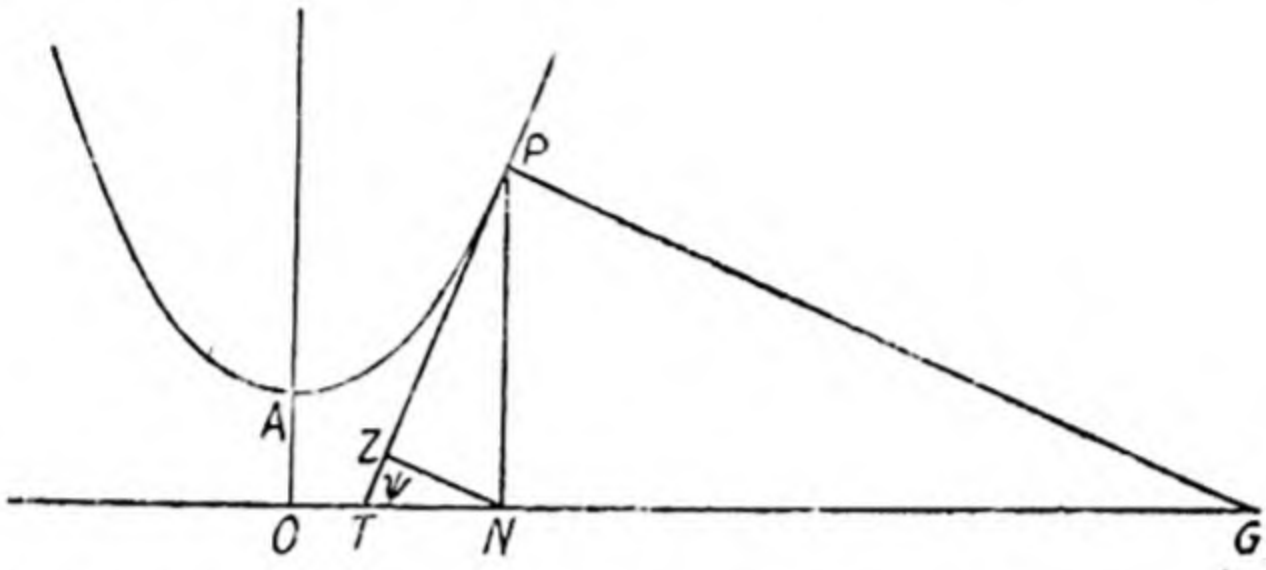
5. Catenary : Tractrix. The catenary is the curve in which a uniform heavy chain hangs freely under gravity. From elementary statical principles it follows that if s be the arc of the curve measured from the lowest point, say A, up to any point P, and ψ the inclination to the horizontal of the tangent at P, then $s=a\tan\psi$, where a is a constant.

If x, y be the co-ordinates of P with respect to some horizontal and vertical co-ordinate axes, we have

$dx/d\psi=dx/ds.ds/d\psi=\cos\psi.a\sec^2\psi=a\sec\psi$ and
$dy/d\psi=dy/ds.ds/d\psi=\sin\psi.a\sec^2\psi=a\tan\psi\sec\psi.$

From these two equations it follows that

$x=a \log \tan (\pi/4+\psi/2)+c_1$, and $y=a \sec \psi+c_2$, where c_1 and c_2 are any arbitrary constants.



If the origin is so determined that $x=0$, $y=a$, for $\psi=0$, that is, at a distance a vertically below A, then $c_1=0$, $c_2=0$.

Now we have $x/a=\log \tan (\pi/4+\psi/2)$

$$=\log (\sec \psi+\tan \psi),$$

whence $\sec \psi+\tan\psi=e^{\frac{x}{a}}$

and therefore $\sec \psi-\tan\psi=e^{-\frac{x}{a}}$.

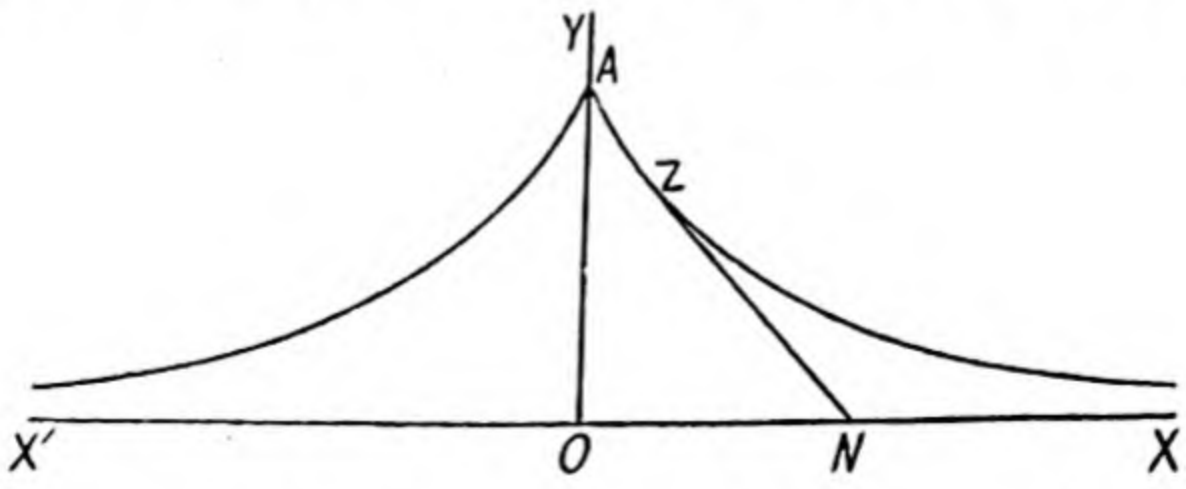
Hence $y=a \cosh x/a$.

A well known property of this curve is that NZ is constant and equal to a.

The curve can be easily traced from the equation

$$y=a \cosh x/a.$$

The locus of the point Z is called the **tractrix** or **Tractory**.



It is easy to verify that its simplest parametric equations are $x=a\ (\cos t+\log \tan t/2)$, $y=a \sin t$.

It can also be easily seen that the portion of the tangent intercepted between the curve and the x-axis is of constant length a.

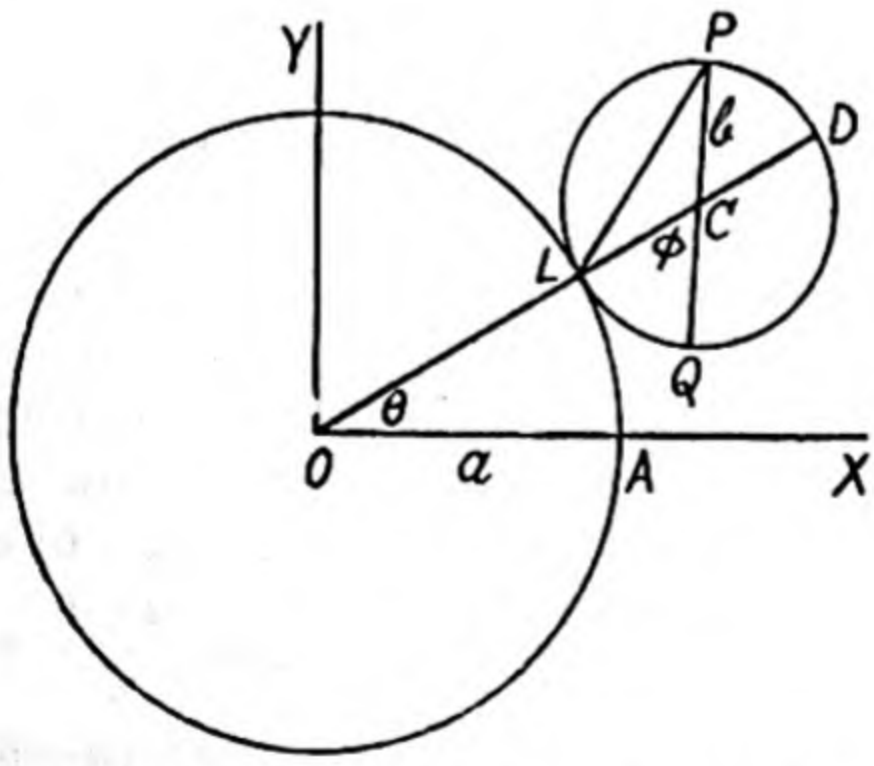
The envelope of normals of the tractrix is the catenary itself.

It may be remarked that the name Tractrix is derived from the fact that it is the path of a heavy particle dragged along a rough horizontal plane by a string, the other end N of which is made to describe a straight line OX.

6. Epicycloids and Hypocycloids.

The curve traced out by a point on the circumference of a circle which rolls in contact with a fixed circle is called an epicycloid or a hypocycloid according as the rolling circle is outside or inside the fixed circle. Those epicycloids in which the rolling circle surrounds the fixed circle may be called as pericycloids when a distinction is desired.

Let O be the centre of the fixed circle, C that of the rolling circle in any position, L the point of contact and P the tracing point. Suppose further that initially the other extremity Q of the diameter PCQ is in contact with A. Let us take the case in which the two circles are external to each other.



Let OA $=a$, CP $=b$, $\angle$ LOA $=\theta$ and $\angle$ LCQ $=\phi$.

The inclination of CP to OA then is $\theta+\phi$.

Let us take OA as the axis of x and O as origin and let P be (x, y).

Then x=Projection of OP on OX
=Projection of OC on OX+Projection of CP on OX
$=(a+b)\cos\theta+b\cos(\theta+\phi)$
Similarly y
=Projection of OC on OY+Projection of CP on OY
$=(a+b)\sin\theta+b\sin(\theta+\phi)$ .... (*i*)
But since $a\theta$=arc AL=arc QL=$b\phi$,

therefore $x=(a+b)\cos\theta+b\cos\frac{a+b}{b}\theta$

$$y=(a+b)\sin\theta+b\sin\frac{a+b}{b}\theta \qquad \text{.....(ii)}$$

The epicycloid traced by Q can be found by changing the sign of the last terms in the expressions for x and y because the projections of CP and CQ are equal and opposite in sign.

The equations therefore for the locus of Q are

$$x=(a+b)\cos\theta-b\cos\frac{a+b}{b}\theta$$

$$y=(a+b)\sin\theta-b\sin\frac{a+b}{b}\theta \qquad \text{..... (iii)}$$

This has a cusp at every point where the tracing point comes into contact with the fixed circle.

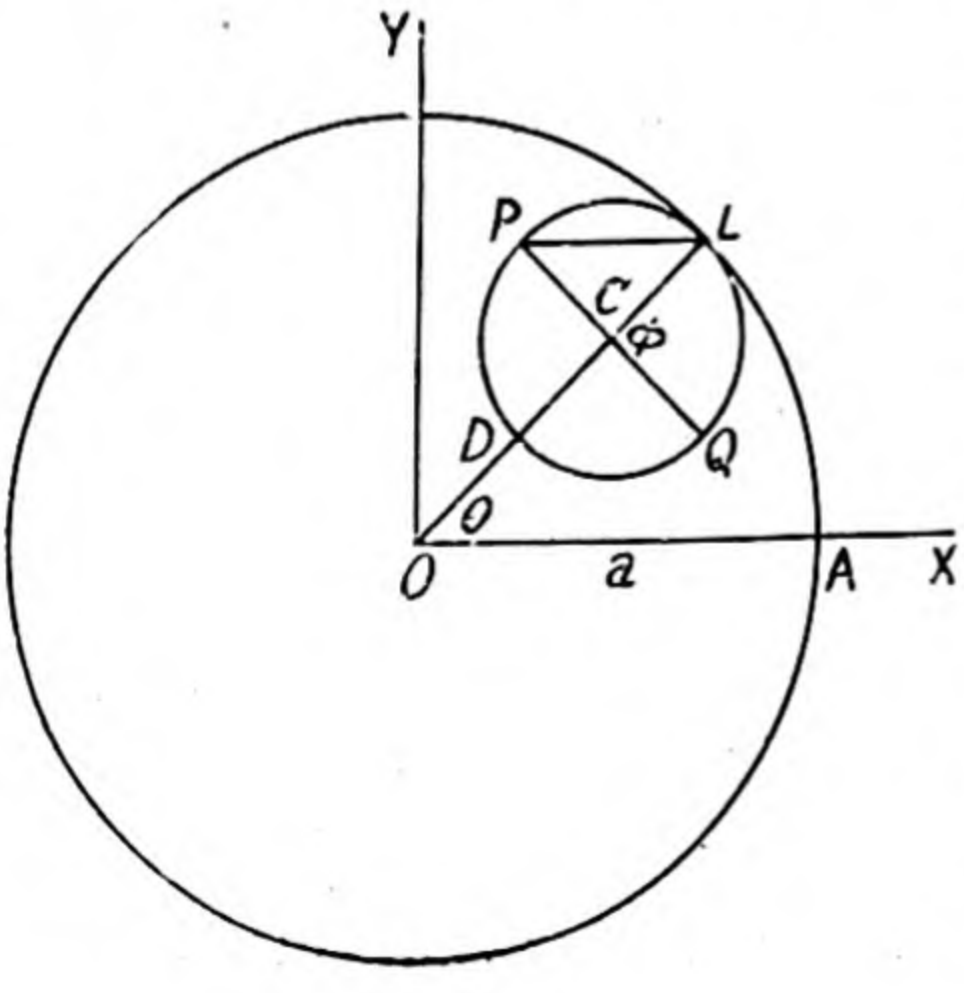
In the above case the circles lie on the opposite sides of the tangent at L. If they lie on the same side, as is the case in pericycloids and hypocycloids, we have only to change the sign of b throughout in (*ii*) and then we get for this case $x=(a-b)\cos\theta$

$$-b\cos\frac{a-b}{b}\theta,$$

$$y=(a-b)\sin\theta$$

$$+b\sin\frac{a-b}{b}\theta \qquad \text{.....(iv)}$$

which can be verified directly by a figure.



In the hypocycloids $a > b$ and in the pericycloids $a < b$.

Similarly for the locus of Q we get from (*iii*)

$$x=(a-b)\cos\theta+b\cos\frac{a-b}{b}\theta$$

$$y=(a-b)\sin(\theta-b)\sin\frac{a-b}{b}\theta. \qquad \ldots\ldots(v)$$

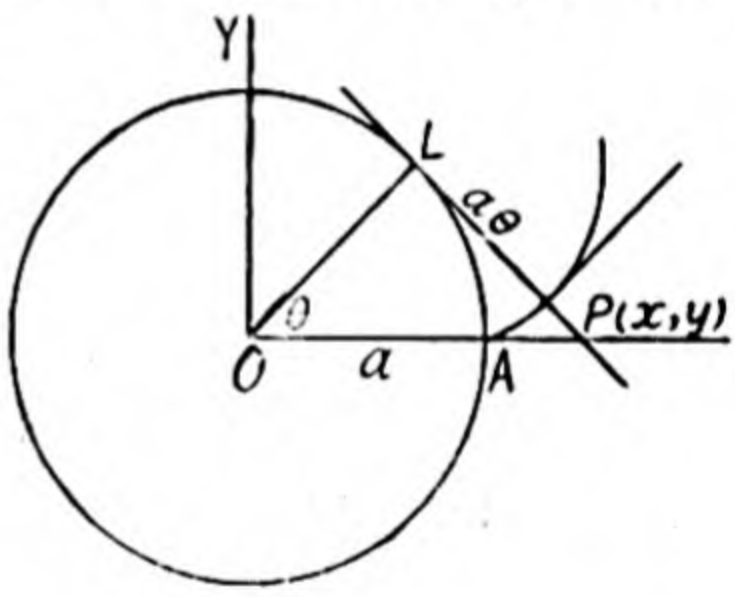
The pedal equation of the epicycloid is

$$r^2=a^2+4b(a+b)/(b+2b)^2.\ p^2$$

and that of the hypocycloid can be obtained by changing the sign of b.

Special Cases. (*a*) If the radius of the fixed circle be infinitely great, we go back to the case of a cycloid. The corresponding equations may be obtained from equations (*i*) above by writing $x+a$ for x and $a\theta=b\phi$ and then finally $\theta=0$.

(*b*) If the radius of the rolling circle is infinite, we get the locus of a point P of a straight line which rolls on a fixed circle. The corresponding equations may be obtained as limiting forms of equations (*iii*) above or may be directly obtained from a figure. They are



$$x=a\cos\theta+a\theta\sin\theta$$

$$y=a\sin\theta-a\theta\cos\theta \ldots(vi)$$

(*c*) If the radius of the rolling circle be taken equal to that of the fixed circle we get a cardioid ; for setting $a=b$ in equation (2) above, we get

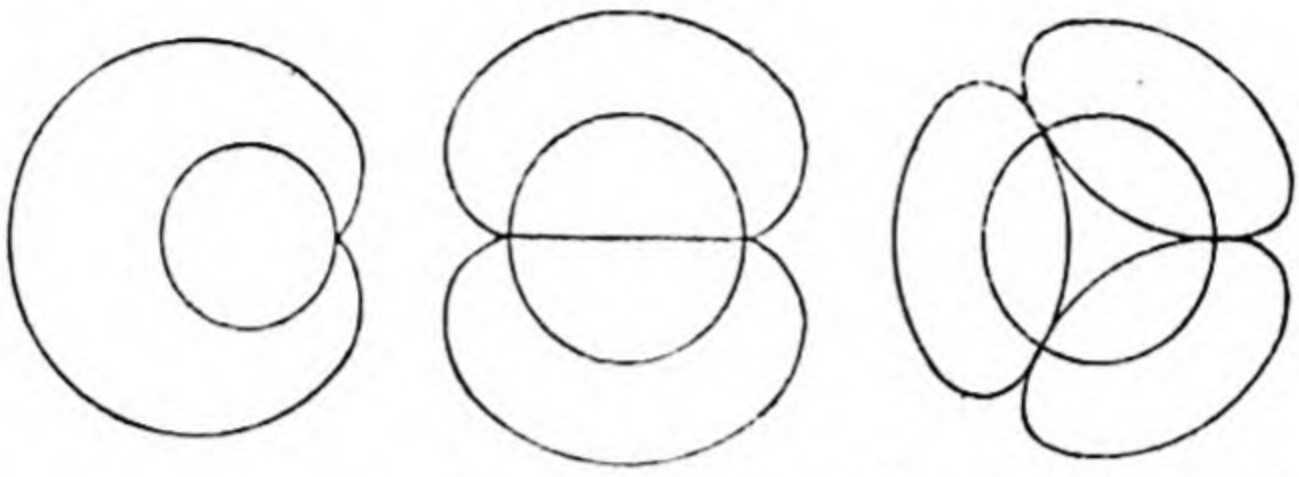
$$x=2a\cos\theta+a\cos 2\theta,\ y=2a\sin\theta+a\sin 2\theta\ ;$$

that is, $x+a=2a(1+\cos\theta)\cos\theta$, $y=2a(1+\cos\theta)\sin\theta$, which shows that the radius vector drawn from the point $(-a, 0)$ as pole is given by $r=2a(1+\cos\theta)$.

This fact can be easily verified from a figure.

It may be noticed that if the radii a and b be commensurable, then after some exact number of revolutions the tracing point will return to its original position and subsequent to this it will repeat its previous path. In such cases the curve is algebraic because the trigonometric functions can be eliminated

between the two equations for x and y. The following figures show the epi- and hypocycloids in which the ratio of the rolling circle to that of the fixed circle has the values 1, $\frac{1}{2}$, $\frac{1}{3}$ respectively.



The curve traced out by a point of the rolling circle which is not on the circumference is called an **epitrocoid** or a **hypotrocoid** as the case may be. If h be distance of the tracing point from the centre of the rolling circle the equations for x and y in the various cases can be obtained by writing h instead of b is only the co-efficients of the second terms. For example corresponding to equations (*ii*) above we get

$$x=(a+b)\cos\theta+h\cos(a+b)\,\theta/b$$
$$y=(a+b)\sin\theta+h\sin(a+b)\theta/b.$$

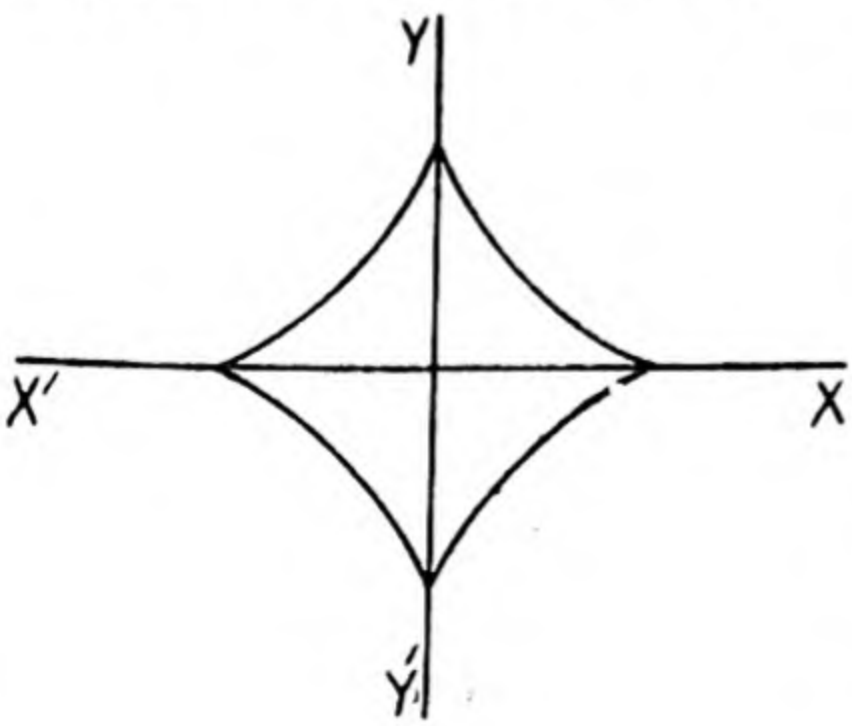
Ex. If in the equations (*v*) above we take $b=a/4$, we get

$$x=\tfrac{3}{4}a\cos\theta+a/4\cos 3\theta=a\cos^3\theta$$
$$y=\tfrac{3}{4}a\sin\theta-a/4\sin 3\theta=a\sin^3\theta.$$

Eliminating θ from these equations, we get

$$x^{\frac{2}{3}}+y^{\frac{2}{3}}=a^{\frac{2}{3}}.$$

This curve can be easily traced from its parametric equations $x=a\cos^3\theta$, $y=a\sin^3\theta$. It has got four cusps and is called the four-cusped hypocycloid, also sometimes called astroid. Its well known property is that the portion of the tangent intercepted between the co-ordinate axes is constant.



7. Polar co-ordinates. The Limacon.

Let O be a fixed point on a circle and Q any other point on it. If on OQ points P, P′

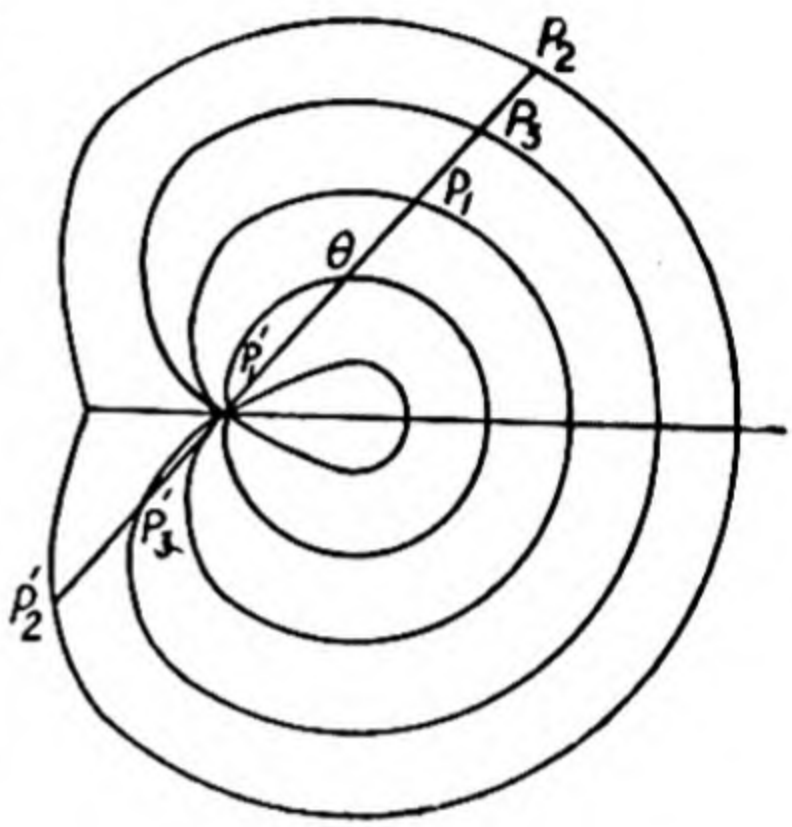
be taken such that P, P′ are at equal constant distance b from Q, the locus of these points is called a limacon.

Take O as pole and the diameter through O as the initial line ; also let the radius of the circle be $a/2$.

The equation of the limacon is evidently

$$r=a\cos\theta\pm b.$$

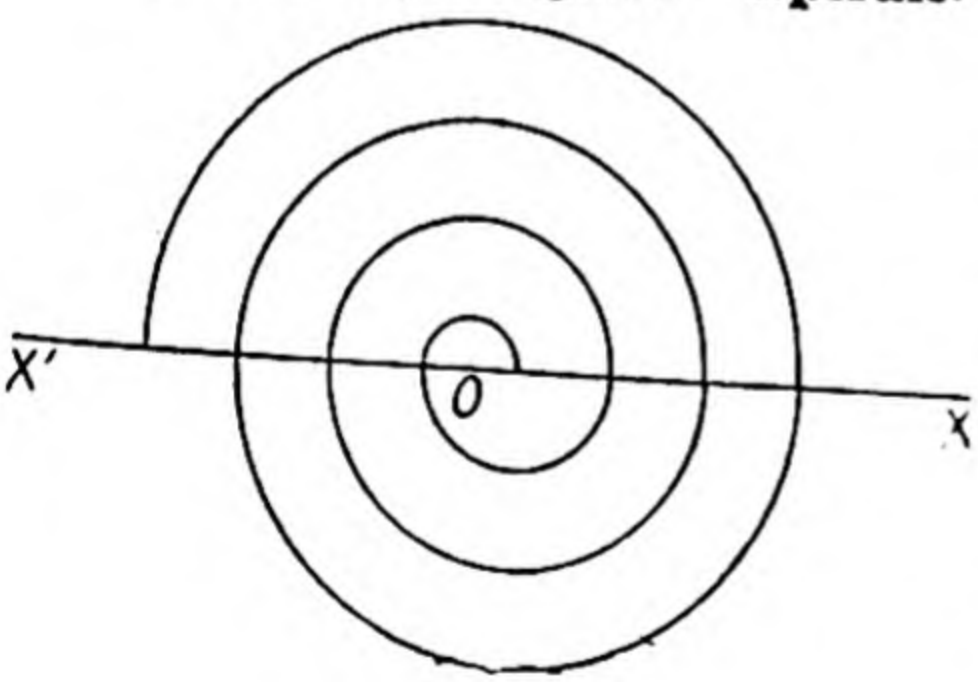
If $b<a$, the curve passes through O when $\theta=\cos^{-1}(-b/a)$, and forms a loop. See the curve traced out by the points P_1, P'_1 in the above figure. If $b>a$, r cannot vanish ; see the curve traced out by the points P_2 and P'_2.



If $b=a$, we get a cusp at the pole. The locus is now called a cardiod and its equation is $r=a(\cos\theta\pm1)$; see the curve traced out by the points P_3 and P'_3.

The Spirals

8. (*a*) Equiangular Spirals. The equiangular spiral is defined by the property that the curve makes a constant angle with the radius vector.



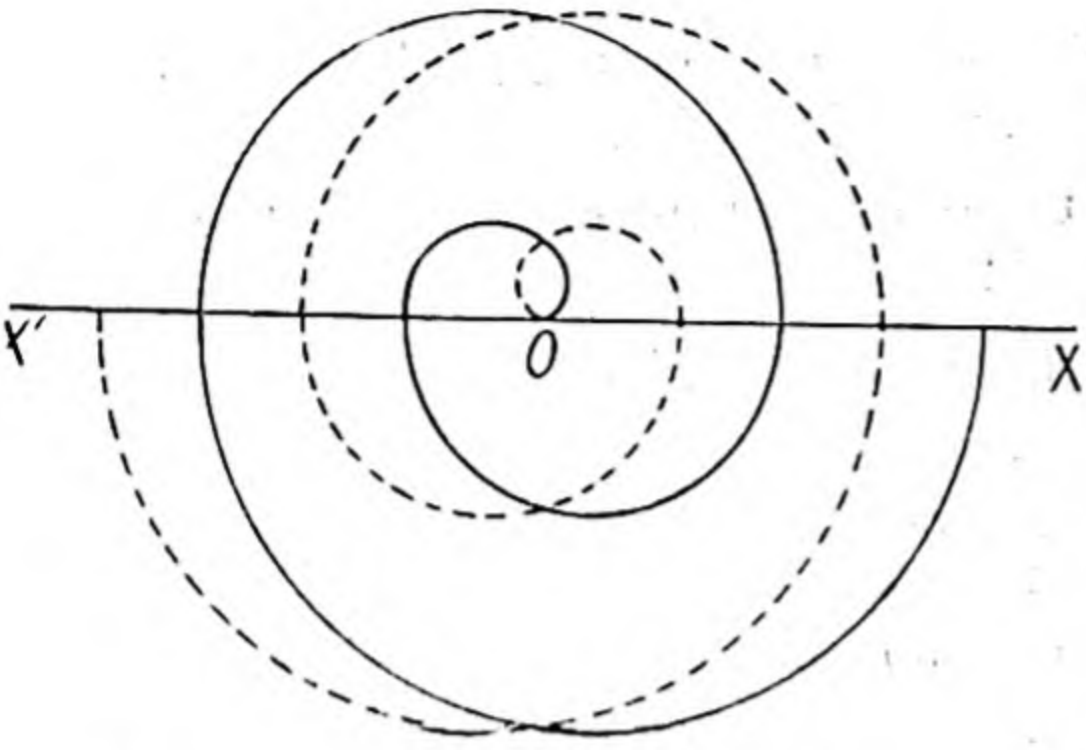
If the constant angle is α, we have

$r\,d\theta/dr=\tan\alpha$ or $dr/r=\cot\alpha\,d\theta$, from which it follows that $\log r=\theta\cot\alpha+c$, where c is an arbitrary constant.

Hence $r=e^{\theta\cot\alpha}e^{c}$

Or $r=a\,e^{\theta\cot\alpha}$, where $a=e^{c}$.

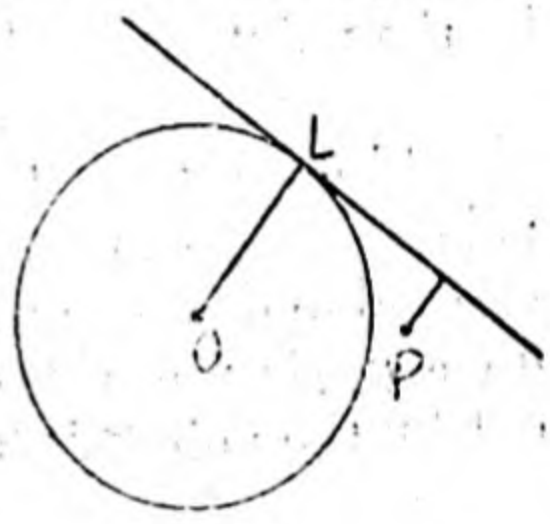
As θ increases from 0 to ∞, r also increases from a to ∞. The shape of the curve is as shown in the figure.



The Spiral of Archimedes. The curve $r=a\theta$ is called the Spiral of Archimedes.

As θ increases, r also increases. The curve is shown in the adjacent figure. The dotted curve corresponds ot negative values of θ.

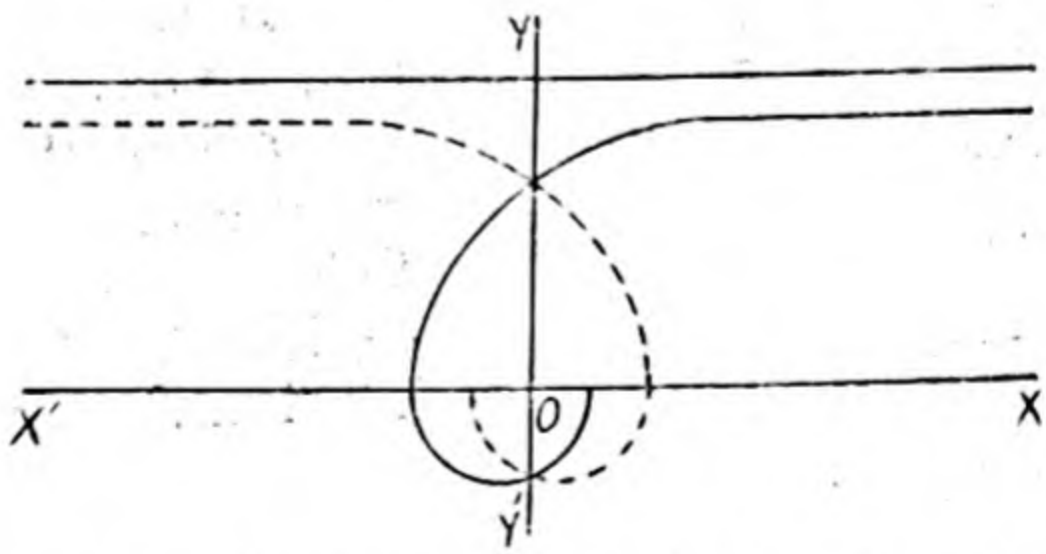
This curve can be generated in several ways. For example it is described by a point which travels along a straight line with constant velocity, whilst the line rotates with constant angular velocity about a fixed point in it. Or, again, it is the locus of the point P relatively fixed with regard to a line which rolls on a fixed circle, as shown in the figure.



(*c*) **The Reciprocal Spiral** is defined by the equation $r=a/\theta$.

As θ approaches 0, r approaches infinity and $r \sin \theta = a$ is an asymptote to the curve.

The dotted branch corresponds to negative values of θ.



(*d*) **The Sine Spiral** is defined by the equation $r^n=a^n \cos n\theta$. This includes as special cases a large number of well known curves.

For example, if $n=\pm 1$, we get the circle $r=a\cos\theta$ and the straight line $r\cos\theta=a$.

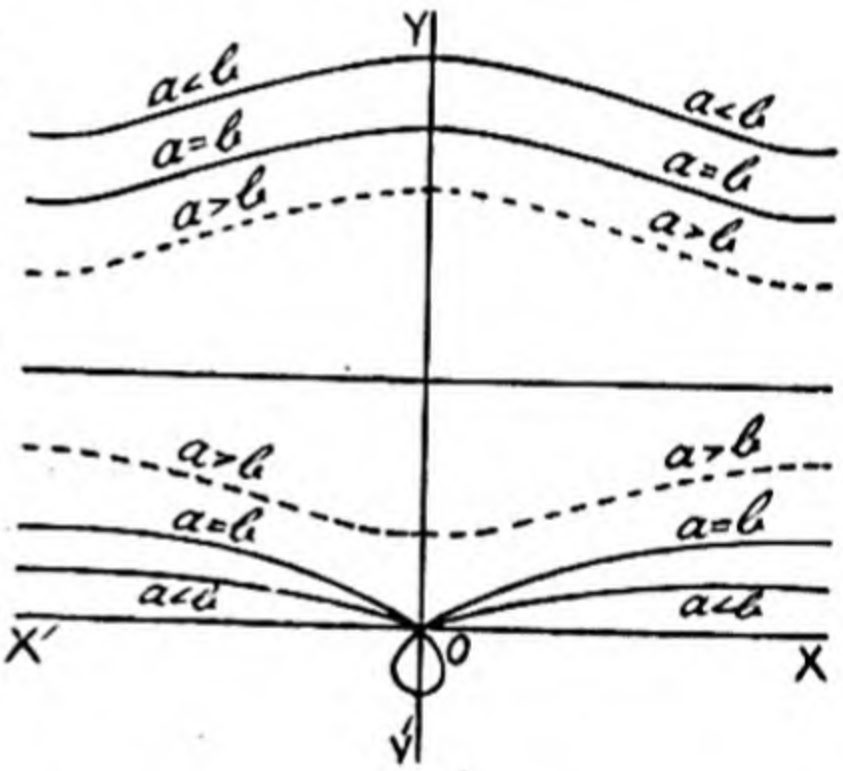
If $n=\pm 2$, we get the Lemniscate of Bernoulli $r^2=a^2\cos 2\theta$ and the rectangular hyperbola $r^2\cos 2\theta=a^2$.

If $n=\pm 1/2$, we get the cardioid $r=a/2\,(1+\cos\theta)$ and the parabola $r=2a/(1+\cos\theta)$

The curves $r^n\cos n\theta=a^n$ and $r^n=a^n\cos n\theta$ are inverse of each other.

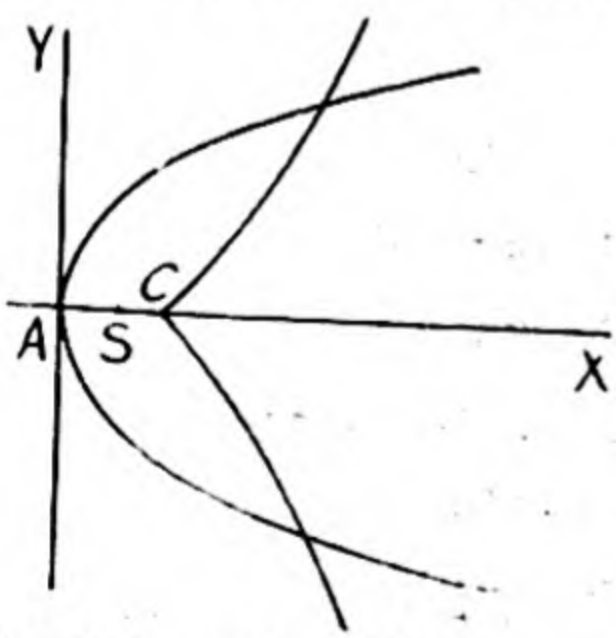
The pedal equation of $r^n=a^n\cos n\theta$ is $p\,a^n=r^{n+1}$.

9. Conchoid of Nicomedes. This curve is defined by the equation $r=a\operatorname{cosec}\theta\pm b$.



The curve consists of two branches. The Cartesian equation is $(x^2+y^2)(y-a)^2=b^2y^2$, and includes both the branches. There is symmetry about the y-axis or the line $\theta=\pi/2$. Tangents at the origin are $a^2x^2+(a^2-b^2)y^2=0$.

Thus the origin is a node, a cusp or a conjugate point according as $a<b$, or $a=b$ or $a>b$. In the first case there is a loop below the x-axis. Also, the line $y=a$ or $r\sin\theta=a$ is an asymptote, the two branches lying on its oppposite sides. There are two or four points of inflexion according as $b>a$ or $b<a$.



An angle can be trisected by means of this curve.

10. Evolute of a Parabola. The evolute of $y^2=4ax$ is

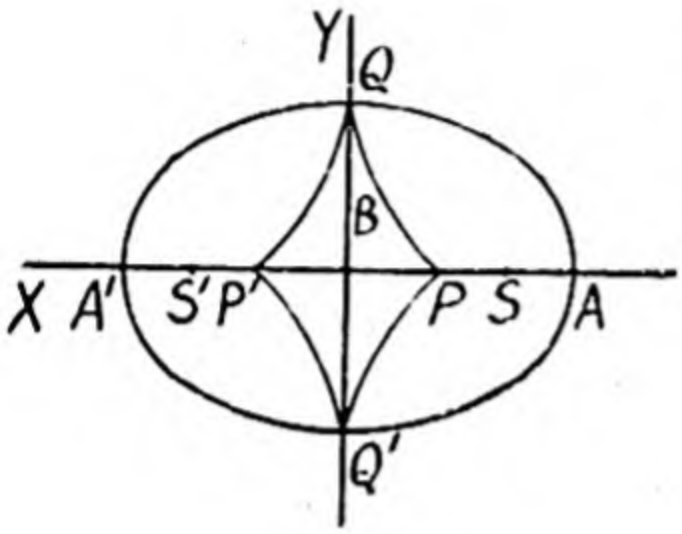
$$27ay^2=4(x-2a)^3.$$

The cusp is at $(2a,\ 0)$ and the curve cuts the parabola again at

$$x=8a.$$

11. Evolute of Ellipse. The evolute of

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



is $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

There is a cusp at each point where the curve meets the co-ordinate axes.

Radii of curvature at A and B are $\frac{b^2}{a}$ and $\frac{a^2}{b}$ respectively.

Thus

$$AP=\frac{b^2}{a},\ BQ=\frac{a^2}{b^2}$$

Length of evolute is

$$4\left(\frac{a^2}{b}-\frac{b^2}{a}\right).$$

EXAMPLES XLVI

Trace the following curves :—

1. $y^2x=a^2(a-x)$.
2. $y^2(a-x)=x^2(a+x)$.
3. $ay^2=x(a-x)$.
4. $a^2x^2=y^2(a^2-y^2)$.
5. $a^4x^2=y^4(a^2-y^2)$.
6. $a^2y^4=x^4(a^2-x^2)$.
7. $a^4y^2=x^5(2a-x)$.
8. $y^3=2ax^2-x^3$.
9. $yx^2=1$.
10. $x^2(1-y)y=1$.
11. $y^2x=x^2+1$.
12. $y^2x=x^2-1$.
13. $y^2x^2=x^2+1$.
14. $y^2x^2=x^2-1$.
15. $y^2(a^2-x^2)=a^3x$.
16. $x^5+y^5=2a^3xy$.
17. $x^5+y^5=5ax^2y^2$.
18. $x^2(a^2-x^2)=8a^2y^2$.
19. $y^2(a+x)=x^2(3a-x)$.
20. $x^2(x+y)=y^2$.
21. $3ay^2=x(x-a)^2$.
22. $y^2(a^2+x^2)=x^2(a^2-x^2)$.
23. $x^2(y+1)=y^2(x-4)$.
24. $x^4+y^4=4ax^2y$.
25. $(x^2+y^2)^2=4axy^2$.
26. $x^4=a^2(x^2-y^2)$.
27. $y^2(a-x)=x^2(a+x)$.
28. $xy^2=a^2(a-x)$.

29. $y^2=(x-1)(x-2)(x-3)$.

30. $y^2=(x-a)(x-b)(x-c)$ where $0<a<b<c$.
considering the form when (*i*) $a=b$ (*ii*) $b=c$ (*iii*) $a=b=c$.

31. $(x^2+y^2)^2=a^2x^2+b^2y^2$.

32. $y^2(x-2a)=x^3$. **33.** $(x/a)^{\frac{2}{3}}+(y/b)^{\frac{2}{3}}=1$.

34. $x=a\sin 2t$, $y=a\sin t$.

35. Show that the curve $xy+1=y^3$ has only one point of inflexion and that it is situated at the intersection with the y-axis. Trace the curve.

36. $y=2x^3-3x^2-12x+5$.

37. $x^4-x^2y+y^3=0$. **38.** $y^2(a^2-x^2)=a^3x$.

39. $x^2y+xy^2+xy+y^2+3x=0$, showing that no curve lies between $x=-4$, $x=-1$; $x=0$, $x=3$.

40. $2x(y-3)^2=3y(x-1)^2$. **41.** $x^2(x+y)=x+4y$.

42. $r^2\cos\theta=a^2\sin 3\theta$. **43.** $r=a\sin 2\theta$.

44. $r=a\cos 3\theta$. **45.** $r\cos\theta=a\cos 2\theta$.

46. $r\cos\theta=a\sin 3\theta$. **47.** $r=a(\sec\theta+\cos\theta)$.

48. $r=3+2\cos\theta)$. **49.** $r=2+3\cos\theta$.

50. $r=a\theta\cos\theta$.

51. Trace the curves $r=a\sin 3\theta$ and $a=r\sin 3\theta$ with regard to the same axes.

52. $r=2a\cos^2\theta$.

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ANSWERS

EXAMPLES I

1. 5, 3 ; not attained. **2.** 8 (attained), 2 (not attained.) **3.** 1 (not attained), $\frac{1}{2}$ (attained). **4.** $\frac{1}{2}$ (attained), 0 (not attained). **5.** 3, 2 ; not attained.

EXAMPLES II

1. 4. **2.** $\frac{5}{3}$. **3.** a^2. **4.** $\frac{n}{m}a^{n-m}$. **5.** 0.

6. ∞ if $n>m$, a/a_1 if $n=m$; 0 if $n<m$, n being supposed to be positive.

7. $\frac{\log a}{\log b}$. **8.** (*i*) $\cos x$ (*ii*) $\sec x \tan x$.

10. (*i*) $\frac{2}{5}$, (*ii*) $(a/b)^k$, (*iii*) 8. **12.** $\frac{1}{2}$. **13.** $\frac{3}{2}$. **14.** $\frac{1}{2}$.

EXAMPLES III

1. (*i*) Single-valued, implicit, algebraic. (*ii*) many-valued, explicit, transcendental.

2. (*i*) $y=+\sqrt{x^2-4}$, $y=-\sqrt{x^2-4}$; $|x|<2$.
(*ii*) $y=\cos^{-1}(4+2x)$, $(-\frac{3}{2}.-\frac{5}{2})$.

3. (*i*) $x=y^{\frac{1}{3}}$ $[-\infty,\infty]$, three
(*ii*) $\sin^{-1}y$, $|y|<1$, unlimited.

6. (*i*) Yes. (*ii*) No. Not for $x=-1$ and $x=2$.

7. (*i*) Discontinuous. (*ii*) Discontinuous.

8. $x=1, 4$.

EXAMPLES IV

1. $-\frac{3}{x^4}$. **2.** $\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}$. **3.** $\frac{x-\sqrt{1+x^2}}{\sqrt{1+x^2}}$.

4. $x^{m-1}(1-x)^{n-1}\{m-(m+n)x\}$.

5. $\frac{2}{n}x(a^2+x^2)^{\frac{1-n}{n}}$. **6.** $\frac{a^2}{(a^2+x^2)^{\frac{3}{2}}}$.

7. $(2x-1)(1-x+x^2)^{-\frac{1}{2}}\quad(1+x-x^2)^{-\frac{3}{2}}$. **8.** $\dfrac{2ab}{a^2-b^2x^2}$.

9. $\dfrac{1}{\sqrt{x^2+a^2}}$. **10** $\dfrac{1}{2x+}-\dfrac{4}{2x+3}-\dfrac{x^2}{x^2-5}$.

11. $\dfrac{1-\log 2x}{x^2}$. **12.** $2\times5^{2x-5}\log 5$.

13. (*i*) $a^x x^a\left(\log a+\dfrac{a}{x}\right)$.

(*ii*) $a^{x^n}\, nx^{n-1}\log a$. (*iii*) $x_{n-1}e^{ax}\log x\left(n+ax+\dfrac{1}{\log x}\right)$.

14. $\dfrac{1}{2(1+x)}\log_{10}e$. **15.** $\dfrac{1}{2}\dfrac{1+2x}{(1+x+x^2)\log(1+x+x^2)}$.

EXAMPLES V

1. $2\sin x\cos x$. **2.** $-m\sin mx$.

3. $2x\sin x+x^2\cos x$. **4.** $\dfrac{1}{1+\sin 2x}$. **5.** $-\sin x$.

6. $-m\cos^{m-1}x\sin x$.

7. $m\sin^{m-1}x\cos^{n+1}x-n\cos^{n-1}x\sin^{m+1}x$.

8. $nx^{n-1}\cos x^n$.

9. $-\sin(\sin x)\cos x$. **10.** $\dfrac{\cos\sqrt{x}}{2\sqrt{x}}$. **11.** $-\dfrac{1}{x}\sin(\log x)$.

12. $2\cos 2x\sin 2x$.. **13.** $\sec x$.

14. $-\sec x$. **15.** $\operatorname{cosec} x$. **16.** $e^x(\tan x+\sec^2 x)$.

17. $e^{ax}(a\sin bx+b\cos bx)$. **18.** $\dfrac{2ab}{a^2\cos^2 x-b^2\sin^2 x}$.

19. $\dfrac{2x}{1+x^4}$. **20.** $\dfrac{1}{1+x^2}$. **21.** $\dfrac{1}{\sqrt{2x-x^2}}$.

22. $\dfrac{ab}{a^2\cos^2x+b^2\sin^2x}$. **23.** $\dfrac{2}{1+x^2}$. **24.** $\dfrac{2-xy}{x^2}$.

25. $\dfrac{y\sec^2 x-\sin y}{x\cos y-\tan x}$. **26.** $\dfrac{k\tan^{k-1}x\sec^2 x}{m\tan^{m-1}y\sec^2 y}$.

EXAMPLES VI

3. $\dfrac{-2(1+2\cos x\cosh x)}{(\sinh x+\sin x)^2}$, $\dfrac{3}{1-x^2}$, $\dfrac{2}{1-x^2}$.

4. $\tanh x, \dfrac{\operatorname{sech}^2 x}{\tanh x}$, $-\coth x$, $a^{\sinh x}\cosh x, \log a$.

5. $\cosh^3 x, \dfrac{1}{\sqrt{x^2-1}}$.

6. $-\dfrac{x}{\sqrt{1-x^2}(1-\sqrt{1-x^2}}$ $-\dfrac{1}{x}$, $\dfrac{x}{(1+\sqrt{1+x^2})\sqrt{(1+x^2)}}$ $-\dfrac{1}{x}$.

7. $\dfrac{1}{x}$, $2\operatorname{cosec} h\, 2x$.

8. $\dfrac{1}{\sqrt{a^2+x^2}}$. Read $\log(x+\sqrt{x^2+a^2})$ for $\log(x\sqrt{x^2+a^2})$.

EXAMPLES VII

1. $\dfrac{y\tan x+\cot x}{\log\cos x}$. 2. $x^x(1+x+x\log x)$.

3. $\dfrac{20}{(x-1)^2}-\dfrac{180}{(x-2)^2}+\dfrac{420}{(x-3)^2}-\dfrac{280}{(x-4)^2}$.

4. $\frac{1}{2}\sqrt{x}\sqrt{\sin x}\sqrt{\log x}\left\{\dfrac{1}{x}+\cot x+\dfrac{1}{x\log x}\right\}$.

5. $(\tan x)^{\cot x}\operatorname{cosec}^2(1-\log\tan x)$.

6. $(\sin x)^{\log x}\left(\dfrac{x\cot x\log x+\log\sin x)}{x}\right)$.

7. $\dfrac{x^{\sin x}}{x}(\sin x+x\cos x\log x)$.

8. $(\sin^{-1}x)^x\left\{\log\sin^{-1}x+\dfrac{x}{\sqrt{1+x^2}\sin^{-1}x}\right\}$

9. $(\sec x)^{\operatorname{cosec} x}\{\sec x-\operatorname{cosec} x\cot x\log\sec x\}$

$+(\operatorname{cosec} x)^{\sec x}(\sec x\tan x\log\operatorname{cosec} x-\operatorname{cosec} x)$.

10. $\dfrac{\log\tan y-y\sec^2 x\cot x}{\log\tan x-x\sec^2 y\cot y}$.

MISCELLANEOUS EXAMPLES

1. $\dfrac{x^2-1}{x^2}$. **2.** $\dfrac{cb-ad}{(c+dx)^2}$. **3.** $\dfrac{1}{2\sqrt{x}}\cos\sqrt{x}$.

4. $bnx^{n-1}\cos(a+bx^n)$.

5. (*i*) $\dfrac{\pi}{180}\cos x^\circ$ (*ii*) $\dfrac{\pi}{180}\sec x^\circ \tan x^\circ$.

6. $pqx^{q-1}\sin^{p-1}x^q\cos x^q$. **7.** $\dfrac{2x}{\sqrt{1-x^4}}$.

8. (*i*) $-\dfrac{1}{1+x^2}$. (*ii*) $\dfrac{2a^2x}{a^4+x^4}$.

9. (*i*) $-\dfrac{1}{x\sqrt{x^2-1}}$. (*ii*) $\dfrac{\sin\alpha}{1-2x\cos\alpha+x^2}$.

(*iii*) $\dfrac{e^x\cos e^n}{1+\sin e^x}$. **10.** $\operatorname{sech} 2x$.

11. $-\dfrac{\sqrt{b^2-a^2}}{b+a\cos x}$. **12.** (*i*) $\frac{11}{8}x^3$. (*ii*) $-\cot x$.

13. $\dfrac{1+\cos x}{-\sin x}$. **14.** 1. **15.** $\frac{3}{4}$. **16** (*i*) $\frac{1}{2}$. (*ii*) 1.

17. (*i*) $-\dfrac{b}{a}\cos\theta$. (*ii*) $-\dfrac{b}{a}\tan\phi$. (*iii*) $\tan t$.

18. $\dfrac{-3}{\sqrt{1-x^2}}$. **19.** $\frac{1}{2}$.

20. $\dfrac{1}{2\sqrt{x}(1+x)}$. $\dfrac{1}{1+x^2}$. **21.** $\frac{1}{2}$.

22. $\dfrac{1}{\sqrt{1-x^4}}$. **23.** $\dfrac{-x}{\sqrt{1-x^2}(1+\sqrt{1-x^2})}-\dfrac{1}{x}$.

24. $\dfrac{1-2x}{2\sqrt{1-x^2}}$. **25.** $\dfrac{1+x^2+x^2\log x}{x\sqrt{1+x^2}}x\sqrt{1+x^2}$.

26. $(\operatorname{cosec} x+\log\sin x\sec x\tan x)(\sin x)^{\sec x}$.

$$\frac{\cos^2x-\sin^2x\log\sin x}{\sin x}(\sin x)^{\cos x}+$$

$$\frac{\cos^2x\log\cos x+\sin^2x}{\cos x}(\cos x)^{\sin x}.$$

28. $\dfrac{b}{2ay+ab}$. **29.** $\dfrac{y}{2y-x}$. **30.** $\dfrac{\sec^2 x}{2y-1}$.

31. $\dfrac{y^2 \tan x}{y \log \cos x - 1}$. **32.** $\dfrac{y^2}{x(1-y \log x)}$.

34. $-\dfrac{yx^{y-1}+y^x \log y}{x^y \log x + xy^{x-1}}$.

35. $e^{x^2}(1+x^2)^{-\frac{3}{2}}\{x(1+2x^2)\tan^{-1} x+1\}$.

36. $\dfrac{x^2-1}{x^4-4}$. **37.** $\dfrac{1}{(1-x)\sqrt{x}}$. **38.** $\dfrac{1}{x\sqrt{x+1}}$.

39. $ae^{at}\cos(b \tan^{-1} t) - \dfrac{b}{1+t^2} e^{at} \sin(b \tan^{-1} t)$.

41. $\dfrac{\pi}{2}$, $-\dfrac{\pi}{2}$. **42.** Continuous but not differentiable.

43. (*i*) $2x \cos x^2$. (*ii*) $\frac{1}{2}x^{-\frac{1}{2}} e^{\sqrt{x}}$. (*iii*) $2x(x^2+3)^{-2}$.
(*iv*) $\cot x$.

EXAMPLES VIII

1. $\frac{5}{2}$. **3.** $[-\infty, 1], [2, \infty]$; $[1, 2]$. **13.** $-\frac{1}{4}, \frac{3}{2}$. $-\frac{9}{4}, 0$.

14. $[\pm(\frac{7}{3})^{\frac{1}{2}}, \pm(\frac{7}{3})^{\frac{3}{2}}]$.

16. $-\dfrac{B}{A} = \dfrac{f(a+h)-f(a)}{\phi(a+h)-\phi(a)}$. **18.** Not applicable.

EXAMPLES IX

1. $\dfrac{(-1)^n \lfloor n}{(1+x)^{n+1}}$. **2.** $\dfrac{\lfloor n}{(1-x)^{n+1}}$. **3.** $\dfrac{b^n \lfloor n}{(a-bx)^{n+1}}$.

4. $(-1)^n \dfrac{\lfloor n+2}{(x+a)^{n+3} \lfloor 2}$.

5. $(-1)^{n+1} \dfrac{1.3.5.....(2n-3)}{2^n} \dfrac{1}{(x+a)^{n-\frac{1}{2}}}$.

6. $\dfrac{1}{2} \cos\left(x+\dfrac{n\pi}{2}\right) - \dfrac{3^n}{2} \cos\left(3x+n\dfrac{\pi}{2}\right)$.

7. $\frac{1}{4}\left\{6^n \cos\left(6x+\frac{n\pi}{2}\right)+4^n \cos\left(4x+\frac{n\pi}{2}\right)\right.$
$\left.+2^n\left(\cos 2x+\frac{n\pi}{2}\right)\right\}$

8. $\frac{1}{8}\{4^n \cos(4x+\frac{1}{2}n\pi)+2^{n+2}\cos(2x+\frac{1}{2}n\pi)\}$.

9. $\frac{e^x}{2}\left\{1+5^{\frac{n}{2}}\cos(2x+n\tan^{-1}2)\right\}$.

10. $\frac{e^x}{2}\left\{2^{\frac{n}{2}}\cos\left(x+\frac{n\pi}{4}\right)-10^{\frac{n}{2}}\cos(3x+n\tan^{-1}3)\right\}$

11. $\frac{e^{2x}}{16}\left\{2.5^{\frac{n}{2}}\cos\left(x+n\tan^{-1}\frac{1}{2}\right)\right.$

$\left.-13^{\frac{n}{2}}\cos\left(3x+n\tan^{-1}\frac{3}{2}\right)-29^{\frac{n}{2}}\cos\left(5x+n\tan^{-1}\frac{5}{2}\right)\right\}$

12. $x^2(3\log x+1)$, $x(6\log x+5)$, $6\log x+11$, $6/x$.

20. $-\frac{1}{a(1-\cos\theta)^2}$.

21. $\frac{3(1-\cos t)}{2(\sin 2t-\sin t)^2}$.

EXAMPLES X

1. $(-1)^n \underline{|n}\left\{\frac{1}{3}\frac{1}{(x-2)^{n+1}}+\frac{5}{3}\cdot\frac{1}{(x+1)^{n+1}}\right\}$.

2. $(-1)^n \underline{|n}\left\{\frac{1}{4}\frac{21}{(x-3)^{n+1}}-\frac{1}{4}\frac{1}{(x+1)^{n+1}}\right\}$.

3. $(-1)^n \underline{|n}\left\{\frac{1}{3}\frac{1}{(1-x)^{n+1}}+\frac{1}{(x+2)^{n+1}}\right\}$.

4. $\frac{(-1)^n \underline{|n}\ 3n}{2}\left\{\frac{1}{(3x-1)^{n+1}}-\frac{1}{(3x+1)^{n+1}}\right\}$.

5. $(-1)^n \underline{|n}\left\{\frac{1}{(x-1)^{n+1}}+\frac{1}{(x+1)^{n+1}}-\frac{1}{(x+2)^{n+1}}\right\}$.

6. $(-1)^n \underline{|n}\ a^n \sin(n+1)\theta \sin^{n+1}\theta$, where $\cot\theta=ax$.

7. $\frac{(-1)^n \underline{|n}}{a^2-b^2}\left[\frac{\sin(n+1)\phi \sin^{n+1}\phi}{b^{n+2}}\right.$

$\left.-\frac{\sin(n+1)\theta \sin^{n+1}\theta}{a^{n+2}}\right]$ where $x=a\cot\theta=b\cot\phi$.

8. $\dfrac{(-1)^n \lfloor n}{4a^3}\left[\dfrac{1}{(x-a)^{n+1}}-\dfrac{1}{(x+a)^{n+1}}-\dfrac{2}{a^{n+1}}\right.$

$\left.\sin(n+1)\theta \sin^{n+1}\theta\right]$ where $x=a\cot\theta$.

9. $\dfrac{1}{4}\dfrac{(-1)^n \lfloor n}{(1+x)^{n+1}}+\dfrac{1}{4}\dfrac{\lfloor n}{(1-x)^{n+1}}-\dfrac{(-1)^n \lfloor n}{2}\sin(n+1)\theta$,

$\sin^{n+1}\theta$ where $\theta=\tan^{-1}\dfrac{1}{x}$.

10. $\dfrac{(-1)^n 2^{n+2} \lfloor n}{3^{\frac{n+2}{2}}}\sin(n+1)\theta \sin^{n+1}\theta$,

where $x=\frac{1}{2}(\sqrt{3}\cot\theta-1)$.

11. $\dfrac{(-1)^n 2^{n+2} \lfloor n}{3^{\frac{n+2}{2}}}\sin(n+1)\theta \sin^{n+1}\theta$, where

$x=\frac{1}{2}(\sqrt{3}\cot\theta+1)$.

12. $\dfrac{(-1)^n 2^{n+1} \lfloor n}{3^{\frac{n+2}{2}}}\{\sin(n+1)\theta \sin^{n+1}\theta-\sin(n+1)\phi$

$\sin^{n+1}\phi\}$ where $2x=\sqrt{3}\cot\theta+1=\sqrt{3}\cot\phi-1$.

13. $\dfrac{(-1)^n \lfloor n}{3}\left\{\dfrac{1}{(x+1)^{n+1}}-\dfrac{(x-2)2^{n+2}\sin(n+1)\theta \sin^{n+1}\theta}{3^{\frac{n+2}{2}}}\right.$

$\left.+\dfrac{2^{n+1}}{3^{\frac{n+1}{2}}}\sin n\theta \sin^{n}\theta\right\}$ where $2x=\sqrt{3}\cot\theta+1$

14. $\frac{1}{2}(-1)^n \lfloor n\left\{\dfrac{1}{(x+1)^{n+1}}-(x-1)\sin(n+1)\theta \sin^{n+1}\theta\right.$

$\left.+\sin n\theta \sin^{n}\theta\right\}$ where $x=\cot\theta$.

15. $\dfrac{(-1)^n \lfloor n}{4}\left\{\dfrac{(n+1)(n+2)}{(x-1)^{n+3}}+\dfrac{3(n+1)}{(x-1)^{n+2}}+\dfrac{1}{2}\dfrac{1}{(x-1)^{n+1}}\right.$

$\left.-\dfrac{1}{2(x+1)^{n+1}}\right\}$.

16. $\dfrac{(-1)^n \lfloor n}{a^{n+1}} \cos(n+1)\theta \sin^{n+1}\theta$. **17.** $\dfrac{(-1)^n \lfloor n}{2a^{n+1}} \sin^{n+1}\theta$

$[\sin(n+1)\theta-(n+1)\cos(n+2)\theta \sin\theta]$.

18. $\dfrac{(-1)^{n-1} \lfloor n-1}{a^n} \sin n\theta \sin^n\theta$ where $x=a \cot\theta$.

19. $2(-1)^{n-1} \lfloor n-1 \sin^n\theta \sin n\theta$ where $x=\cot\theta$.

20. $\dfrac{(-1)^{n-1} \lfloor n-1}{(\sin\alpha)^{n+1}} \sin n\theta \sin^n\theta$ where $x=\cos\alpha+\sin\alpha \cot\theta$.

EXAMPLES XI

1. $e^x(x^2+2nx+n^2-n)$. **2.** $a^{n-1}e^{ax}(ax+n)$.

3. $x^3\cos(x+n\pi/2)+3nx^2\cos[x+(n-1)\pi/2]$
$+3n(n-1)x\cos[x+(n-2)\pi/2]+n(n-1)(n-2)$
$\times\cos[x+(n-3)\pi/2]$.

4. $(-1)^{n-1}\dfrac{2 \lfloor n-3}{x^{n-2}}$. **5.** $\dfrac{(-1)^{n-2}e^x \lfloor n-1}{x^n}$.

$\left(1-\dfrac{nx}{n-1}+\dfrac{nx^2}{(n-2) \lfloor 2}\cdots+\dfrac{(-1)^{n-1}nx^{n-1}}{\lfloor n-1}+(-1)^{n-1}\log x\right)$

6. $(-1)^{n-1} \lfloor n-3 \sin^{n-2}\theta\{(n-2)(n-1)\sin n\theta\cos^2\theta$
$-2n(n-2)\sin(n-1)\theta\cos\theta+n(n-1)\sin(n-2)\theta\}$,
where $x=\cot\theta$.

7. $(a^2+b^2)^{\frac{n}{2}}e^{ax}\sin\left(bx+n\tan^{-1}\dfrac{b}{a}\right)+n(a^2+b^2)^{\frac{n}{2}}$

$\times e^{ax}\sin\left[bx+(n-1)\tan^{-1}\dfrac{b}{a}\right]$.

9. $\frac{1}{4}\sin 2x-32\sin 4x$.

11. $(x^2-1)y_{n+2}+x(2n+1)y_{n+1}=0$.

13. If $n=2k$, $y_n=0$; if $n=2k+1$, $y_n=(-1)^k m(m^2-1^2)$
$\times(m^2-3^2)\ldots\ldots[m^2-(2k-1)^2]$.

18. $\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\ldots\ldots(m^2-2^2)\,m^2$, n even ;
$[m^2-(n-2)^2]\{m^2-(n-4)^2\}\ldots\ldots(m^2-1^2)\,m$, n odd.

23. $a=2$, $n=0$, 1.

EXAMPLES XII

1. $-\frac{x}{(x^2+y^2)^{\frac{3}{2}}}, -\frac{y}{(x^2+y^2)^{\frac{3}{2}}}$.

2. $\frac{y}{x^2+y^2}, -\frac{2xy}{(x^2+y^2)^2}$; $-\frac{x}{x^2+y^2}, \frac{2xy}{(x^2+y^2)^2}$.

3. $me^{m(x+y)}, m^2e^{m(x+y)}$; $me^{m(x+y)}, m^2e^{m(x+y)}$.

4. $e^y \cos x, -e^y \sin x$; $e^y \sin x, e^y \sin x$.

11. $2x \tan^{-1}\frac{y}{x} - y$, $2\tan^{-1}\frac{y}{x} - \frac{2xy}{x^2+y^2}, \frac{x^2-y^2}{x^2+y^2}$.

14. $-3/2$.

EXAMPLES XIII

1. $-\frac{x}{\sqrt{a^2-x^2}}, -\frac{a^2}{(a^2-x^2)^{\frac{3}{2}}}$.

2. $\frac{x^2-ay}{y^2-ax}, \frac{2a(x^2-ay)(y^2-ax)-2y(x^2-ay)-2x(y^2-ax)^2}{(y^2-ax)^2}$.

3. $\frac{\sin B \cos C+\sin A \cos C-\sin C \cos A-\cos A \sin B}{\sin C \cos B-\sin B \cos C-\sin A \cos C+\sin A \cos B}$.

5. $\left(\frac{\partial f}{\partial z}\cdot\frac{\partial \phi}{\partial x}-\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial x}\right)\Big/\left(\frac{\partial \phi}{\partial z}\frac{\partial f}{\partial y}-\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial y}\right)$.

6. $\frac{3(1-2x^2z)}{4y(z-3)}\cdot\frac{1-6x^2}{2(z-3)}$.

7. $\frac{z-x}{y-z}, \frac{x-y}{y+z}$.

8. $(i)\ -y$, $(ii)\ -\frac{y}{x^2+y^2}$, $(iii)\ 0$.

10. 0.

EXAMPLES XIV

2. $\cos B\delta a+\cos A\delta b+a \sin B\delta c$.

8 0 84 sq. in.

EXAMPLES XV

15. $\frac{\partial u}{\partial x}=\frac{x+3v^2+3v^2x^2}{xy}, \frac{\partial u}{\partial y}=-\frac{3v^2+ux}{xy}$,

$\frac{\partial v}{\partial x}=-\frac{3x^2+v}{x}, \frac{\partial v}{\partial y}=\frac{1}{x}$.

EXAMPLES XVI

11. $2x-\frac{2x^3}{3}+\frac{2x^5}{5}-\ldots$ **13.** $\theta+\frac{\theta^2}{\lfloor 2}-\frac{5}{\lfloor \underline{4}}\theta^4+\ldots\ldots$

17. $1+x-\frac{2x^3}{\lfloor 3}+\ldots+\frac{2^{\frac{n}{2}}}{\lfloor n}\,x^n\cos\left(n\frac{\pi}{4}\right)+\ldots\ldots$

$$x+x^2+\frac{x^3}{3}+\ldots+\frac{2^{\frac{n}{2}}}{\lfloor n}\,x^n\sin\left(n\frac{\pi}{4}\right)+\ldots\ldots$$

18. $\frac{x^2}{\lfloor 2}+\frac{2^2}{\lfloor 4}+\frac{2^2.4^2}{\lfloor 6}x^6+\ldots\ldots$

19. $x-\frac{x^3}{3}+\frac{x^5}{5}-\ldots$

EXAMPLES XVII

1. max. for $x=4$, min for $x=8$.
2. min. for $x=7$; max. for $x=1$.
3. max. for $x=-1$; min. for $x=\frac{7}{9}$; neither for $x=3$. **4.** 6.
10. $y=11$ min. **11.** max. at $x=1$, min. at $x=2$.
13. $x=-2, -1, 1, 2$, give max. min. alternately, values being $-16, -38, 38, 16$.
14. for $x=1$, max. for $x=5$ min.
15. max. at $x=2$; min. at $x=2\frac{6}{11}$. **16.** 2 max., -2 min.
18. $\frac{x^2-x+1}{x^2+x-1}$. **24.** 1 hour 39 mts., 4 miles from P.
25. $a/4$, $3a/4$.
30. (*i*) $x=n\pi+\pi/6$ gives a minima, $x=n\pi+5\pi/6$ gives a minima ; (*ii*) $x=\frac{k\pi}{n+1}$, where k is any integer gives maxima and minima alternately beginning with $k=1$; omitting when n is even the case when $k=0$ or a multiple of $n+1$. (*iii*) $\theta=\tan^{-1}\frac{\sqrt{3}}{2}, \frac{\pi}{2}$,

$$\pi-\tan^{-1}\frac{\sqrt{3}}{2}, \pi+\tan^{-1}\frac{\sqrt{3}}{2}, \frac{3\pi}{2}, 2\pi-\tan^{-1}\frac{\sqrt{3}}{2}\ldots\ldots$$

give max. and min. alternately.
(*iv*) If $a>b$ max.$=a$, min.$=b$. If $a<b$, max.$=b$, min.$=a$.
31. (*i*) max. for $x=(a+2b)/3$. (*ii*) max. for $x=a$.
If $a=b$, neither max. nor min.
33. **When AP and PB make equal angles with OX.**

EXAMPLES XIX

1. max. 54, min. 50, greatest 70, least 0. **2.** 1, 21.

EXAMPLES XX

1. $y=2$, max. **2.** Neither. **3.** $y=1$, max.
4. $(4a, 4a)$. **9.** $u=6$ max., $u=1$, min.
10. (*i*) ab/b. max. (*ii*) $\frac{a^2b^2}{a^2+b^2}$, min. **13.** 24 mts. ; 6 miles.

14. $\tan\theta$ is (*i*) $\left(\frac{y_1}{x_1}\right)^{\frac{1}{3}}$, (*ii*) $\left(\frac{y_1}{x_1}\right)^{\frac{1}{2}}$, (*iii*) $\frac{y_1}{x_1}$.

15. $\frac{a}{\sqrt{2}}$. **16.** $\sqrt{\frac{40}{3}}$ ft. in width and length and

$\frac{10}{\sqrt{30}}$ ft. in depth.

17. $\frac{1}{2}$. **20.** $\frac{2\pi c^3}{9\sqrt{3}}$. **22.** $\frac{l}{3(a-b)}$. **27.** $2\pi[1-\sqrt{\frac{2}{3}}]$.

31. (*i*) Length 2 ft., girth 4 ft. (*ii*) Length $1\frac{3}{4}$ ft. girth $4\frac{1}{4}$ ft.

EXAMPLES XXI

1. 4. **2.** $\frac{1}{4}$. **3.** $-\frac{1}{2}$. **4.** $\frac{1}{\sqrt{2}}$. **5.** $\frac{a}{b}$. **6.** $\frac{\log a}{\log b}$.

7. $\frac{1}{6}$. **8.** $-\frac{a}{3}$. **9.** 1. **10.** -1. **11.** $1+\log a$.

12. $\frac{1}{2}$. **13.** ∞. **14.** ∞. **15.** $\pi^2/2e$. **16.** $\frac{1}{4}$. **17.** $-\frac{1}{2}$.

EXAMPLES XXII

1. 1. **2.** 0. **3.** 1. **4.** 1. **5.** 0 **6.** a.

EXAMPLES XXIII

1. $\frac{2}{3}$. **2.** $\frac{1}{3}$. **3.** $e^{-\frac{b^2}{2a^2}}$. **4.** 1. **5.** $1/e$. **6.** $e^{\frac{1}{3}}$.

7. ae. **8.** 1 ; $e^{-\frac{1}{2}}$; 0 or ∞ according as x tends to 0 through +ve or −ve values.

EXAMPLES XXIV

2. 1. **3.** 1. **4.** $\frac{1}{3}$. **5.** 1. **6.** 0. **7.** $\frac{1}{8}\pi^2$.

8. $\frac{1}{2}$. **9.** $\frac{1}{\sqrt{e}}$. **10.** 0. **11.** 0. **12.** $2/\pi$.

13. a. **14.** $\frac{1}{2}$. **15.** $\pi^2/6$. **16.** 1.

17. 1. **18.** $\frac{2}{5}$. **19.** $-\frac{e}{2}$. **21.** 4.

22. 1. **23.** $\frac{3}{2}$ **24.** [Differentiate $w.\ r.$ to m x, and n being treated as contsants] $\sec^2 nx \operatorname{cosec} 2nx$.

25. (*i*) $1+\cos mx$. (*ii*) ∞. **26.** $\frac{2f'''(x)}{3f''(x)}$.

27. If $n>m$, ∞ ; $n=m$; $\frac{A}{a}$; $n<m$, 0.

28. $a=-2$; limit $=-1$. **29.** $a_1=-4$; $a_2=5$; limit $=1$.

EXAMPLES XXV

1. (*i*) $X(2xy+y^2)+Y(x^2+2xy)=3b^3$; $(X-x)(x^2+2xy)$
$=(Y-y)(y^2+2xy)$.

(*ii*) $Xy+Yx=2k^2$; $(X-x)x=(Y-y)y$.

(*iii*) $Y-y=\sinh\frac{x}{a}(X-x)$; $(Y-y)+(X-x)\operatorname{cosech}\frac{x}{a}=0$.

(*iv*) $X\{2x(x^2+y^2)-a^2x\}+Y\{2y(x^2+y^2)+a^2y\}$
$-a^2(x^2-y^2)=0$;
$(Y-y)(2x^3+2xy^2-a^2x)=(X+x)(2yx^2+2y^3+a^2y)$

(*v*) $x^4X+y^4Y=a^5$; $(Y-y)x^4=(X-x)y^4$.

2. (*i*) $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1$; $ax\sec\theta-by\operatorname{cosec}\theta=a^2-b^2$.

(*ii*) $yt=x+at^2$; $y=-tx+2at+at^3$.

(*iii*) $x\sin\frac{\theta}{2}-y\cos\frac{\theta}{2}=a\theta\sin\frac{\theta}{2}$;

$x\cos\frac{\theta}{2}+y\sin\frac{\theta}{2}=a\theta\cos\frac{\theta}{2}+2a\sin\frac{\theta}{2}$.

(*iv*) $y\cos\theta-x\sin\theta+a\theta=0$, $\sin\theta y+x\cos\theta-a=0$,

(*v*) $y\cos 2\theta-x\sin 2\theta+4\sin\theta=0$; $y\sin 2\theta-x\cos 2\theta$
$=2\cos\theta$.

5. $y=2\pm\frac{1}{\sqrt{3}}$. **6.** $y=a^2x$, $y=m(3m+2a)x-2m(m+a)^2$

$y=m(3m-2a)x+2m(m-a)^2$; $(2a, 2a^3)$

$\{-2m,-2m,(2m+a)^2\}$, $\{2m, 2m(2m-a)^2\}$

8. ±1. **9.** $Xx^2+Yy^2=c^3$, $x^2(Y-y)=y^2(X-x)$.

10. $X(3y^2-3y)+Y(-3x+1)+(-3xy+2y-3)=0$.

13. $c=\dfrac{am^{\frac{1}{3}}\left(1-m^{\frac{2}{3}}\right)}{\left(1+m^{\frac{2}{3}}\right)^{\frac{1}{4}}}$ **17.** $\frac{\pi}{4}$.

EXAMPLES XXVI

1. $y=0$. **2.** $x=0$. **3.** $y=0$. **4.** $x=0$.

5. $4y^2+9x^2=0$ the tangents are imaginary. **6.** $x=0$.

7. $3x-2y=0$. **8.** $y=0$. **9.** $y\pm x=0$. **10.** $x+y=0$.

11. $x=0$, $y=0$. **12** $x=0$.

EXAMPLES XXVII

1. $2y=2x-a\pi+4a$, $2y+2x=a\pi$; $\sqrt{2}a.\sqrt{2}a$.

EXAMPLES XXVIII

5. $\frac{\pi}{2}$. **6.** $pa^n=r^{n+1}$. **12.** (*i*) $r^2=2ap$; (*ii*) $p^2=ar$

13. (*i*) $\dfrac{l}{r}=e\cos\theta+\cos(\theta-\theta_1)$. $\dfrac{le\sin\theta_1}{1+e\cos_1\theta\cdot r}\ \dfrac{1}{}=e\sin\theta$

$+\sin(\theta-\theta_1)$

(*ii*) $r_1^2=-ra\sin\theta_1\sin(\theta_2-\theta)+rr_1\cos(\theta_1-\theta)=0$;

$ar_1\sin\theta_1=rr_1\sin(\theta_1-\theta)+ra\sin\theta_1\cos(\theta_1-\theta)=0$

(*iii*) $r_1^2=ra^2\sin2\theta_1\sin(\theta-\theta_1)+rr_1^2\cos(\theta-\theta_1)=0$;

$a^2\sin2\theta_1\,r_1=rr_1^2\sin(\theta_1-\theta)+ra^2\sin2\theta_1\times\cos(\theta-\theta_1)=0$.

(*iv*) $r_1^2=2ar\cos2\theta_1\sin(\theta_1-\theta)+rr_1\cos(\theta_1-\theta)=0$

$2a\cos2\theta r_1=rr_1\sin(\theta_1-\theta)-2a\cos2\theta_1\,r\cos(\theta_1-\theta)$.

11. $2ap=r^2$.

14. (*i*) $\dfrac{b^2}{p^2}=\dfrac{2a}{r}-1$. (*ii*) $\dfrac{b^2}{p^2}=\dfrac{2a}{r}+1$ (*iii*) $p^2=ar$.

EXAMPLES XXX

1. $x-y+1=0$, $x+y+2=0$.

2. $y-x=0$, $y-2x=0$, $y-3x=0$.

3. $y=0$, $y-x+1=0$, $y+x-1=0$.

4. $x-y=0$, $y+x+1=0$, $2y+x+1=0$.

5. $y-x+1=0$, is the only real asymptote.
6. $y-x+1=0$, is the only real asymptote.
7. Same asymptotes as in Q. 3 above.
8. $y=x+2$. 9. $y+\ =0$. 10. $y=x-b/2$, $y=-x-b/2$.
11. $y=-x+2a/3$. 12. $y=2x+1/2$, $y=-2x-1/2$.
13. $x+y=2a/3$.
14. $y=x$, $2y=x$, $3y=x$. 15. $y=x+1$.
16. $x+y+1=0$, $x-y+1=0$, $x+2y+3=0$.
17. $y=x$, $x=\pm(2y-1)$.
18. $x+y-1=0$, $x+3y+1=0$, $3x-y+2=0$.
19. $y\pm x=0$, $2x+3y=0$, $3x+2y=0$.

EXAMPLES XXXI

1. $x=0$, $y=0$. 2. $y=\pm a$, $x=\pm a$. 3. $y=\pm a$.
4. $x=\pm a$. 5. $y+a=0$. 6. None.
7. $x=\pm 1$. 8. None

9. $y=\pm 1$, $x+a=0$. 10. $x=0$, $y=0$.
11. The two real asymptotes are $y=b$, $x+a=0$.
12. $y=\pm 1$, $x=\pm 1$.

EXAMPLES XXXII

1. $y=x$, $x=2y+1$, $x=2y+2$.
2. $y+x=0$, $x-y=0$, $x-y=1$.
3. $y=x\pm 1$, $y+x=\pm 1$. 4. $y=x$, $y=2x+1$, $y=2x+2$.
5. $y=2x$; $y+2x=\pm 2$. 6. $y+x=0$, $y+x-1=0$, $x=0$.
7. $y-x=0$, $y-x-1=0$: $x=0$.
8. $y-x=0$, $y=0$ are the two real asymptotes at finite distances.
9. $x-y+a=0$, $x-y-\frac{a}{2}=0$; $x+a=0$, $x-a=0$;

$$x+y+\frac{a}{2}=0.$$

EXAMPLES XXXIII

1. $x=\pm a$, $y=0$.
2. $y=\pm(x+a,)$ $y=a$. 3. $x=1$, $x=2$, $y=\pm 1$.
4. $x=0$, $y=0$, $y=\pm x$. 5. $x=0$, $y=0$, $y=\pm x$, $x=\pm 3y$.
6. $y+x=1$, $y+x=2$, $x+2y+14=0$, $x+3y=13$.

7. $3x+6y=5$. **8.** $x+y=1$.

9. (*i*) $2x-3y=1$, $2x-3y+3=0$.
(*ii*) $x+2y+2=0$, $y+x\pm2\sqrt{2}=0$.

10. $y-x=2$, $y-x=3$. **11.** $x=0$, $x=y$, $y=x\pm1$.

12. $x+y=\pm2\sqrt{2}$, $x+2y+2=0$. **13.** $x=\pm a$, $y=\pm x$.

14. $y=x$, $y+x+1=0$, $y=2x$, $y+2x+1=0$.

15. $y+x=0$, $y=x-3$, $y=x+2$.

16. $x=2y-14a$, $x=3y+13a$, $x-y=a$, $x-y=2a$

18. $y=0$, $x=\pm1$, $x-y+2=0$.

19. $bxy(x^2-y^2)-a(a^2-b^2)(x^2+y^2-a^2)$.

EXAMPLES XXXV

1. (*i*) $\theta=0$, (*iii*) $2r\sin\theta=\pm a$, $2r\cos\theta+a=0$.

2. $r\cos\theta=a$.

3. $nr\sin(\theta-m\pi/n)=a\sec m\pi$, m being any integer.

5. $2\theta=\pi$, $2r\sin\theta=a$. **6.** $r\cos\theta=a$.

7. $n\theta=m\pi$, m being any integer. **8.** $r\sin\theta=a$.

9. $r\sin(\theta-m\pi/n)=b/n$, m being any integer.

10. $a=2r(\pm\cos\theta-\sin\theta)$. **12.** $r=a$.

13. The two asymptotes are—

$$\left(\frac{el}{r}+(1-e^2)\cos\theta\right)^2=(e^2-1)\sin^2\theta.$$

14. (*i*) $r=c$. (*ii*) $r=a$. (*iii*) $r=a$. (*iv*) $r=1$.

EXAMPLES XXXVI

4. $x=a\sqrt{\dfrac{\sqrt{28}-5}{3}}$. **5.** $\left\{\dfrac{1}{3},\ \pm\dfrac{4}{3\sqrt{3}}\right\}$

6. $x=\frac{1}{2}$. **7.** $x=3$, $\dfrac{28\pm\sqrt{3}}{11}$, pt. of undulation at $x=2$.

12. $\left(\dfrac{1}{2c},\ \dfrac{1}{e^2}\right)$. **17.** $(\sqrt{2}a,\ \frac{1}{2})$.

19. (0, 0) is a single cusp of the first species.

20. (0, 0) is a node.

23. (0, 0) is an isolated point if a, b have the same sign, otherwise a node.

25. $(a, 0)$ is a double point; $\frac{dy}{dx}=\pm a^{-\frac{3}{2}}$.

26. $(0, -1)$, $(1, 0)$, $(-1, 0)$. **27.** $(0, \pm 1)$, $(\pm 1, 0)$.

28. $x=\pm 2$. **29.** $0, \pm\sqrt{b/a}$.

31. (*i*) $(0, 0)$ single cusp, second species.
(*ii*) $(0, 0)$ single cusp, second species.

33. 0, 0 conju. **36.** node at $(1, 2)$, $y=\pm x$.

42. $(-3, 3)$, $(1, -1)$, are double points: $\frac{dy}{dx}=\pm 1$ for both.

43. $x+1=0$, $y+1=0$.

45. $A=\frac{k\pi^2}{a^2(\pi^2-8)}$, $B=\frac{2\cot\beta}{\pi}$, $C=\frac{8k}{8-\pi^2}$.

EXAMPLES XXXVII

1. (*i*) (*a*) $y=mx$; m. (*b*) $y=mx+c$; c.
(*c*) $x\cos\alpha+y\sin\alpha=p$; a.
(*ii*) (*a*) $(x-a)^2+y^2=r^2$; a. (*b*) $x^2+y^2-2ax=0$; a.
(*iii*) $y^2=4(x+\alpha)$; α. For parabolas, read equal parabolas.

2. (*i*) Concentric circles; r. (*ii*) Circles passing through origin with centre on the x-axis; a (*iii*) Family of straight lines touching $x^2+y^2=1$; m (*iv*) Family of straight lines touching the parabola $y^2=4ax$; m.

(*v*) Family of straight lines touching the ellipse or hyperbola; m.

(*vi*) System of co-axal circles; g.

(*vii*) Family of parabolas whose axis is x-axis and whose directrix is the y-axis; a.

(*viii*) Family of confocal ellipses; c.

EXAMPLES XXXVIII

1. (*i*) $y^2=4ax$. (*ii*) $x^2/a^2+y^2/b^2=1$. **2.** $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$.

3. $y+\frac{gx^2}{2u^2}=\frac{u^2}{2g}$. **4.** $(n-1)^{n-1}x^n+n^n cyn^{n-1}=0$.

5. $\frac{2}{x^{2-n}}+\frac{2}{y^{2-n}}=\frac{2}{c^{2-n}}$. **7.** $c^2(x^2+y^2)=x^2y^2$.

8. $\frac{a^4}{y^2}+\frac{b^4}{y^2}=\frac{e^2}{a^2}$.

9. (*i*) $\sqrt{x}+\sqrt{y}=\sqrt{c}$. (*ii*) $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$. (*iii*) $4xy=c^2$.

10. $x^{\frac{2}{3}}+y^{\frac{2}{3}}=c^{\frac{2}{3}}$, c being the given length.

12. $x^{\frac{1}{3}}+y^{\frac{1}{3}}=c^{\frac{1}{3}}$. 13. $27\,ay^2=4(x-2a)^3$.

14. $(x+y)^{\frac{2}{3}}-(x-y)^{\frac{2}{3}}=(4c)^{\frac{2}{3}}$. 16. Two st. lines.

17. $(x^2+y^2)^2=4a^2(x^2-y^2)$.

20. (*i*) $r^{\frac{n}{1-n}}=a^{\frac{n}{1-n}}\cos\frac{n\theta}{n-1}$. (*ii*) $(x-b)^2+y^2=a^2$. (*iii*) Another spiral of the same type.

22. (*i*) $r\cos\theta+a\sin^2\theta$. (*ii*) $r^2=a^2\cos^2\theta+b^2\sin^2\theta$.

(*iii*) $r=2a\cos^3\frac{\theta}{3}$. (*iv*) The auxiliary circle.

(*v*) $r^{\frac{n}{n+1}}=a^{\frac{n}{n+1}}\cos\frac{n}{n+1}\theta$.

EXAMPLES XXXIX

6. $r^m=a^m\cos m\theta$, where $m=\frac{n}{1+kn}$.

7. $r^k=a^k\cos k\theta$, where $k=\frac{n}{1-n}$.

EXAMPLES XLI

1. $a^2x^2+b^2y^2=k^4$. 2. A conic with focus at origin.

3. $2ar=k^2\sec^3\frac{\theta}{3}$. 4. A circle.

5. $r^{\frac{n}{n+1}}=(k^2/a)^{\frac{n}{n+1}}\sec\frac{n}{n+1}\theta$.

6. $x^{\frac{m}{m-1}}+y^{\frac{m}{m-1}}=\left(\frac{k^2}{e}\right)^{\frac{m}{m-1}}$.

EXAMPLES XLIII

1. $\frac{2(a+x)^{\frac{3}{2}}}{a^{\frac{1}{2}}}$, $2a(1+t^2)^{\frac{3}{2}}$. 2. a.

6. $4a\cos\frac{\theta}{2}$, $4a\cos\psi$, $4a\sqrt{\frac{2a-y}{2a}}$.

8. $\frac{(a^4y^2+b^4x^2)^{\frac{3}{2}}}{a^4b^4}$, $(a^2 \sinh^2 u+b^2 \cos h^2 u)^{\frac{3}{2}} / ab$.

9. $x=a\ (4^{\frac{1}{3}}-1)$, $y=\pm 2a\sqrt{4^{\frac{1}{3}}-1}$. 10. $\frac{1}{\sqrt{2}}$.

12. (*i*) $-\frac{2^{\frac{3}{2}}}{6}$. (*ii*) $\frac{2^{\frac{3}{2}}}{6}$. 13. $-3a \sin\phi \cos\phi$, $-3(axy)^{\frac{1}{3}}$

14. (*i*) e^{-x}. (*ii*) $\sqrt{8}$.

EXAMPLES XLIV

1. $\frac{4a}{3}\sin\frac{\theta}{2}$, $\frac{4}{3}\frac{ap}{r}$; $\frac{a^2}{3r}$. 3. $\frac{a}{1+m^2}$.

4. $\frac{a^2}{3r}$. 5. $\frac{a^2}{3r}$. 6. $\frac{a^m}{(m+1)r^{m-1}}$. 7. $-3p$.

8. $-\frac{a^2b^2}{(a^2\cos^2\psi-b^2\sin^2\psi)^{\frac{3}{2}}}$ 9. $\frac{-a^2b^3}{p^3}$. 10. a.

11. $\frac{2pr}{a}$, $2a \operatorname{cosec}^3\psi$. 12. $\frac{a^2b^2}{p^3}$, $\frac{r^3b^2}{ap^3}$, $\frac{a(1-e^2)}{(1-e^2\sin^2\psi)^{\frac{3}{2}}}$.

EXAMPLES XLV

1. $\frac{1}{2}$. 2. $\frac{a}{2}$. 3. $4a$. 4. $\pm 2^{\frac{3}{2}}$. 5. a, $\frac{a}{4}$.

6. $\frac{b^2}{a}$, $\frac{b^2}{c}$. 7. ∞. 8. $-\frac{2\sqrt{2}}{3}$. 9. $\frac{an}{2}$.

10. $\frac{2\sqrt{2a}}{3}$, $2\sqrt{2a}$. 11. $5\sqrt{2}$, $-\frac{5\sqrt{17}}{2}$. 14. $2\sqrt{2}$.

17. $2r$. 24. $-\frac{25}{54}$; $x=-\frac{5}{3}$. 25. (*i*) $-\frac{1}{6}$. (*ii*) ∞.

26. At origin $\rho=1$. 27. (*i*) $\sqrt{2}$, ∞. (*ii*) 0.

29. $\frac{a}{3}$. 31. $\left(\frac{9}{8}, 3\right)$, $\left(\frac{9}{8}, -3\right)$.

32. (*i*) $(2a+3at^2, -2at^3)$; $\left(\frac{a^2-b^2}{a}\cos^3\phi, \frac{b^2-a^2}{b}\sin^3\phi\right)$.

37. $a^2=b^2$; $a^2+2c^2=0$.

EXAMPLES XLVI

1. $(ax)^{\frac{2}{3}}-(by)^{\frac{2}{3}}=(a^2+b^2)^{\frac{2}{3}}$, 8. $\lambda=e^{\frac{\pi}{2}}$. 18. $16\sqrt{3}-9$,